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The last of the Fibonacci groups

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SYNOPSIS

All the Fibonacci groups in the family $F(2, n)$ have been either fully identified or determined to be infinite, bar one, namely $F(2, 9)$. Using computer-aided techniques it is shown that $F(2, 9)$ has a quotient of order 152.5^{741}, and an explicit matrix representation for a quotient of order 152.5^{18} is given. This strongly suggests that $F(2, 9)$ is infinite, but no proof of such a claim is available.

1. INTRODUCTION

Conway [**5**] aroused interest in the Fibonacci groups in 1965. These groups have been studied in general by Johnson, Wamsley and Wright [**10**] and by Chalk and Johnson [**4**]. The Fibonacci group $F(2, n)$ may be presented

$$F(2, n) = \langle x_1, x_2, \ldots, x_n;\ x_1x_2 = x_3, \ldots, x_{n-2}x_{n-1} = x_n,\ x_{n-1}x_n = x_1,\ x_nx_1 = x_2\rangle.$$

Determination of one of these groups was made as early as 1907 [**11**], and by 1974 [**1**] all bar one, namely $F(2, 9)$, had been either fully identified or determined to be infinite. Computer-aided techniques have been used in this investigation of $F(2, 9)$. Computer implementations of group-theoretic algorithms utilized are a coset enumeration program [**3**], a Reidemeister–Schreier program [**7**], a nilpotent quotient algorithm program [**12**], an abelian decomposition program [**9**], and a Tietze transformation program.

All the groups $F(2, n)$ which are known to be finite have been determined and in fact can be identified by coset enumeration [see **8** for details of the most difficult successful coset enumeration]. The group $F(2, 9)$, which is known to have a maximal nilpotent quotient of order 152, resisted all attempts by coset enumeration. The reason for this is now clear. In this paper we show that $F(2, 9)$ has a quotient of order 152.5^{741}. We present the method for the discovery of this quotient, and also give an explicit matrix representation for a quotient of order 152.5^{18}.

The size of the largest quotient that we have been able to discover is governed by the availability of computer resources, and there is every reason to expect that much larger quotients exist. All indications suggest that $F(2, 9)$ is infinite, but proof of such a claim eludes us.

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2. Quotients of $F(2, 9)$ and its Subgroups

The abelian quotient of $F(2, 9)$ is isomorphic to $C_2 \times C_2 \times C_{19}$, where C_n denotes a cyclic group of order n. The nilpotent quotient algorithm shows that $F(2, 9)$ has a maximal nilpotent quotient isomorphic to $Q \times C_{19}$, of order 152 (here Q is the quaternion group). Using this information, it is easy to find presentations for subgroups of indices 2, 4, 8, 19, 38, 76 and 152 in $F(2, 9)$.

We hoped that the nature of the subgroups would cast light on $F(2, 9)$. The simplest hope was that the abelian quotients of the subgroups would provide new information. Using a judicious combination of all the computer programs mentioned in §1 we were able to find the maximal abelian quotients of the subgroups of $F(2, 9)$ corresponding to subgroups of $Q \times C_{19}$.

In each case except the last, namely index 152, the maximal abelian quotient is the same as that of the corresponding subgroup of $Q \times C_{19}$. However the index 152 subgroup of $F(2, 9)$ has a maximal abelian quotient which is elementary abelian of order 5^{18}. The nilpotent quotient algorithm reveals that this subgroup has a maximal 5-quotient of class 3 with order 5^{741}. This shows that $F(2, 9)$ has a quotient of order 152.5^{741}. Some of these computations are further described in [**9**].

The details of the computations outlined above are not particularly perspicuous. (They are available from the authors, as are programs for doing all the calculations.) However, it is possible to distil from the computer calculations a succinct demonstration of the existence of a quotient of $F(2, 9)$ with order 152.5^{18}.

3. A Matrix Representation

In this section we show that $F(2, 9) = \langle x_1, x_2 \rangle$ has a quotient of order 152.5^{18}, which we exhibit as a linear group of dimension 19 over the field $\mathbb{F}_5$ of five elements.

We begin by defining a number of matrices over $\mathbb{F}_5$. Let

$$B=\begin{bmatrix} 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1\\ 1&1&3&1&3&3&3&2&0 \end{bmatrix}\in GL_9(5)$$

(B is the companion matrix of one of the irreducible factors of the polynomial $\lambda^{19}-1$ over $\mathbb{F}_5$); let

$$X_1=\begin{bmatrix} 0 & B^2\\ -B^2 & 0\end{bmatrix},\quad X_2=\begin{bmatrix} 2B^{10} & 0\\ 0 & 3B^{10}\end{bmatrix},$$

so that $X_1, X_2\in GL_{18}(5)$; and for $i=1,2$ let

$$Y_i=\begin{bmatrix} X_i & v_i\\ 0 & 1\end{bmatrix}\in Gl_{19}(5)$$

where

$$v_1=(3,0,3,3,0,0,3,2,1,0,0,0,0,0,0,0,0,0)^T$$
$$v_2=(1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)^T$$

($-^T$ denoting the transposed matrix).

THEOREM. *There is a homomorphism*

$$\phi\colon F(2,9)\to GL_{19}(5)$$

with image of order 152.5^{18}, *and such that*

$$\phi(x_i)=Y_i\ (i=1,2).$$

Proof. We note firstly that since B^2 has order 19, the matrix group $\langle B^2\rangle$ furnishes a 9-dimensional faithful irreducible representation over $\mathbb{F}_5$ of the cyclic group C_{19}. If

$$U_1=\begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix},\quad U_2=\begin{bmatrix} 2 & 0\\ 0 & 3\end{bmatrix}$$

($U_i\in GL_2(5)$), then $\langle U_1, U_2\rangle$ furnishes a 2-dimensional faithful irreducible $\mathbb{F}_5$-representation of the quaternion group Q. Since X_1 and X_2 are, respectively, Kronecker products of B^2 with U_1, and B^{10} with U_2, it follows that $G=\langle X_1, X_2\rangle$ furnishes an 18-dimensional representation of $Q\times C_{19}$, which is faithful because Q and C_{19} have coprime order, and irreducible since its tensor factors have coprime dimension [or, for example, by **6,** Corollary 2.6, where we require the fact that $\mathbb{F}_5$ is a splitting field for Q]. Furthermore, the epimorphisms

$$F(2,9) \to C_{19} \cong \langle B^2 \rangle, \quad x_1 \mapsto B^2, \quad x_2 \mapsto B^{10}$$

and

$$F(2,9) \to Q \cong \langle U_1, U_2 \rangle, \quad x_i \mapsto U_i,$$

give rise to a map

$$F(2,9) \to Q \times C_{19} \cong G, \quad x_i \mapsto X_i,$$

which, it is easy to see, is again an epimorphism.

Let V be an 18-dimensional vector space over $\mathbb{F}_5$, on which $GL_{18}(5)$ is considered to act from the left. In the light of the above, G acts irreducibly on V. The matrix group

$$H = \left\{ \begin{bmatrix} X & v \\ 0 & 1 \end{bmatrix} \in GL_{19}(5): X \in G, v \in V \right\}$$

is isomorphic to the split extension of V by G, and has order

$$|H| = |G| \times |V| = 152.5^{18}.$$

We shall identify V with the (multiplicative) subgroup

$$\left\{ \begin{bmatrix} I & v \\ 0 & 1 \end{bmatrix} : v \in V \right\}$$

of H.

We note that $Y_1, Y_2 \in H$. For $i = 3, 4, 5, \ldots,$ define

$$Y_i = Y_{i-2} Y_{i-1}.$$

It may then be verified by direct calculation that

$$Y_{10} = Y_1, \quad Y_{11} = Y_2.$$

In consequence there is a homomorphism

$$F(2,9) \to H, \quad x_i \mapsto Y_i \quad (i = 1, 2).$$

To complete the proof it remains to be shown that

$$\langle Y_1, Y_2 \rangle = H.$$

The action of $G = \langle X_1, X_2 \rangle$ on V is realized in H via conjugation by the matrices Y_1^{-1} and Y_2^{-1}. It follows that $V \cap \langle Y_1, Y_2 \rangle$ is a submodule of the irreducible $\mathbb{F}_5 G$-module V (it should be remembered that "addition" in the module V is in fact matrix multiplication). A calculation shows that

$$[Y_1^2, \quad Y_2^2] = Y_1^{-2} Y_2^{-2} Y_1^2 Y_2^2 = \begin{bmatrix} I & y \\ 0 & 1 \end{bmatrix},$$

where

$$y = (3, 0, 4, 1, 3, 2, 4, 2, 3, 3, 0, 2, 2, 2, 4, 2, 0, 2)^T \neq 0.$$

Thus $[Y_1^2, \quad Y_2^2]$ is a non-trivial element of $V \cap \langle Y_1, Y_2 \rangle$ so $V \cap \langle Y_1, Y_2 \rangle = V$, that is, $V \leq \langle Y_1, Y_2 \rangle$. Since $\langle Y_1, Y_2 \rangle$ has G as a quotient, we see that its order is divisible by 152.5^{18}. Therefore $\langle Y_1, Y_2 \rangle = H$ as required.

4. Further Calculations

The group theory language Cayley [**2**] was used to investigate the matrix group H described above. Given the two generating matrices Y_1 and Y_2 for H, Cayley confirmed that H has order 152.5^{18}. During the computation permutation representations for H were calculated, including a lowest degree faithful representation which has degree 190.

As a last try we looked at a corresponding index 190 subgroup of $F(2, 9)$. This subgroup has a maximal abelian quotient isomorphic to C_4, which is the same as for the corresponding subgroup of H.

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