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On presentations for unitary groups

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In memory of the long-term inspiration of Charlie Sims

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ABSTRACT

Concise presentations for groups are useful for both theoretical and computational purposes. We give short 2-generator presentations for the 3-dimensional unitary groups $U_3(q)$ and $SU_3(q)$ for q up to 11.

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1. Introduction

Study of presentations for interesting groups dates back to the early days of what was known as abstract group theory. We investigate presentations for small unitary groups. Vital tools used are computer implementations of algorithms devised or improved by Charlie Sims. The 3-dimensional special unitary group $SU_3(q)$ is the cover of the projective special unitary group $U_3(q)$. Unitary groups appeared, as hyperorthogonal, in 1899 in Dickson [9] where it is shown that $U_3(q)$ is simple for $q > 2$, while $U_3(2)$ has order 72.

Concise presentations for the 3-dimensional unitary groups are challenging to produce. The printed version (Conway et al.) of the Atlas [8] gives presentations for many groups but not for 3-dimensional unitary groups. The more recent, online (Wilson et al.) Atlas [23] includes presentations for the five smallest simple ones. We provide short presentations on generating pairs for $U_3(q)$ and $SU_3(q)$ for $q \leq 11$. Most of these are new and include each $U_3(q)$ which appears in the printed Atlas tables.

Unitary groups are not handled in Babai et al. 1997 [1]. Later work has led to short presentations on non-minimal generating sets for general q. Hulpke and Seress 2001 [19] give presentations which are short in theoretical terms. Guralnick et al. 2008 [11] give presentations on 3 generators and at most 21 relators for $SU_3(q)$. They have significantly more relators and they are much longer than ours for small q. In [11] short presentations for individual groups (like ours) are used both in their own right and as building blocks for short presentations for other groups or families of groups.

Leedham-Green and O'Brien [20] have developed Magma code which is used in the Matrix Group Recognition Project. For unitary groups there are presentations on 4 generators and with up to 22 relators.

We use case-inverse notation with generator names: upper-case letters denote inversion so that, for example, $A = a^{-1}$. We give our presentations via sequences of relators (with generator names implicit). We use braces, so that $\{a^2, b^3\}$ denotes a presentation for the modular group defined on the generating sequence (a, b). Most of our presentations are on generators a and b, in which case u is used for ab and v for aB. When a presentation includes only u and v, the generating sequence is (u, v). Generator orders give useful information about a presentation. Since these orders may not be explicitly revealed in the relators of some of our presentations, we say (m, n)-generated to specify orders m and n for the generators in the generating sequence.

Conciseness of a presentation can be measured in various ways including the size of the generating set, the size of the defining set of relators, and the length of relators. Various notions of presentation length are considered in detail in [11, Section 1.2]. Our presentations are on minimal generating sets. We define length to be the sum of the lengths of the relators as words in the group generators and their inverses. Another measure is efficiency, introduced by Epstein in 1961 in connection with the study of 3-manifolds. A presentation is efficient if it achieves the well-known lower bound on the number of relators, which is the number of generators plus the rank of the Schur multiplier. A detailed study of efficient presentations for finite simple groups appeared in 2014 [5].

In Table 1 we provide summary information on a selection of the shortest known presentations on generating pairs. The corresponding presentations appear in two tables in Section 3, where we give further details about the groups and their presentations. In the Notes column of these tables: M means minimal possible length; E means efficient; T means the presentation described in Table 1; and G means good for coset enumeration, CE-good (defined later). The corresponding presentations for Table 1 are identified by T in the later tables.

Table 1
Properties of selected shortest presentations.

Group Name	Group Order	No of Rels	Pres Length	Gen Orders	Total Cosets	Notes
$U_3(2)$	72	3	20	(4,4)	73	MEG
$SU_3(2)$	216	2	16	(4,12)	229	MEG
$U_3(3)$	6048	2	19	(4,12)	198076	ME
$U_3(4)$	62400	2	22	(5,13)	1557671	ME
$U_3(5)$	126000	3	28	(5,4)	1240589	E
$SU_3(5)$	378000	3	29	(4,5)	19731726	
$U_3(7)$	5663616	2	24	(16,16)	20018866	E
$U_3(8)$	5515776	3	36	(21,21)	11300294	E
$SU_3(8)$	16547328	3	39	(7,63)	24855963	G
$U_3(9)$	42573600	3	38	(8,40)	51230621	G
$U_3(11)$	70915680	3	44	(4,20)	86351677	EG
$SU_3(11)$	212747040	4	46	(10,30)	298839543	G

Previously published presentations show that the groups $U_3(2)$, $U_3(3)$, $U_3(4)$, $U_3(5)$ and $U_3(8)$ are efficient. Here we give efficient presentations for $SU_3(2)$, $U_3(7)$ and $U_3(11)$. We give presentations with 2 generators and 3 relators for each of the remaining groups, $U_3(9)$, $SU_3(5)$, $SU_3(8)$ and $SU_3(11)$.

Our results come from computations which use techniques based on procedures described in Sims' book [22]. Many of our presentations come from calculations which rely on the Todd-Coxeter Schreier-Sims algorithm. Since his book, Sims, jointly with some of the authors of this paper, used such ideas in [18,13] to construct short presentations. Our computer calculations use GAP, Magma and some standalone programs. They sometimes use significant resources.

Presentations of a group can vary significantly according to the generating set on which they are defined. Specifically, a relator on one generating set may not be a relator for another. In 1936 Philip Hall [12] described this phenomenon in terms of defining subgroups – that is, kernels of maps from the underlying free group on the generating set. We call these *presentation kernels*. We say that two presentations are *variants* if they have the same presentation kernel. Some of our processes for modifying presentations produce variants while others may change the presentation kernel.

By identifying redundant relators in a presentation and deleting them from it we obtain shorter variants. By amalgamating two relators (replacing them by one which, in the context of the rest of the presentation, implies both) we obtain variants with fewer relators which may also be shorter. Other kinds of relator deletion or amalgamation may define a different group.

We can change generating sets by applying automorphisms of the underlying free group, exemplified in [15]. This way we get presentations which may have different presentation kernels, but they do have the same number of relators. They may vary in length and we target short presentations. More details can be found in Section 2.

We use coset enumeration to confirm that our presentations define groups of interest. Sims [22, Chapters 4, 5, 6 and 7] studies coset enumeration and related topics in depth and provides a solid foundation for our work. We generally use the ACE coset enumerator, as available via GAP [10] and Magma [2], or as a standalone program [17] for some more difficult cases.

In addition to shortness and efficiency, we evaluate presentations from a performance point of view by providing some coset enumeration statistics, along the lines of [5]. Quite different figures can arise from presentation variation and from enumeration strategy changes so, for consistency, we provide the total number of cosets used in a successful enumeration over the trivial subgroup using the `Hard` strategy of ACE4. These enumerations, available via [17], are for performance assessment and are not the best way to verify presentation correctness. We call a presentation *CE-good* if total cosets is less than twice the index, for this strategy choice. We provide short, CE-good presentations for each of our groups.

2. Details

In [7] we observed that $U_3(11)$ arises as a section of a one-relator quotient of the modular group. This quotient has a short presentation which enables us to build short, 2-generator presentations for both $U_3(11)$ and $SU_3(11)$. Since short presentations were not readily available for other unitary groups, we were motivated to find them for $U_3(q)$ and $SU_3(q)$ for q up to 11.

The modular group is relevant to 3-dimensional unitary groups in that only five are not $(2,3)$-generated, namely $U_3(2)$, $SU_3(2)$, $U_3(3)$, $U_3(5)$ and $SU_3(5)$ (see [21]). So all the others are presentable as quotients and one, $U_3(4)$, is known to be a one-relator quotient [3].

In addition to observing that a group is a section of another with a concise presentation, we have two ways for finding short presentations for specific groups. One, coming from Methods 1 and 3 in [4,5], begins with lists of short presentations. The other, Method 2 in [5], starts with generating sets for specific groups.

- Presentation enumeration. We generate and investigate all (up to appropriate equivalence) short presentations which define groups with suitable structure, and check if we have found a presentation for a group of interest. We use canonical forms to provide representative presentations. Notions of equivalence and canonicity are

exemplified in [15,7]. Presentation enumeration is the basis for claims that certain presentations are shortest, cf. [3].
- Generating sets. We consider generating pairs for groups based on matrix or permutation representations, sometimes using selected sets [14] to facilitate the process. We can construct presentations on pairs with specific properties. We use the Todd-Coxeter Schreier-Sims algorithm as implemented by the `Magma` command `FPGroup`, which uses the ideas of [6], to obtain initial presentations on the chosen generators.

Once we have a presentation we try to reduce the number of relators or shorten the presentation by relator deletion and by relator amalgamation. We use various tools, especially coset enumeration, to investigate the presentations which arise.

We also use Tietze transformation-based approaches. We have a standalone program `automac` (automorphic Andrews-Curtis) which is based on ACME (Andrews-Curtis Move Enumerator) [16]. It combines length-preserving automorphisms of the extended symmetric group [15] and some Whitehead automorphisms with Andrews-Curtis moves.

Many effective instances of presentation manipulation by amalgamation of power relators with other relators appear in the literature. Some applications appear in [5] together with a theorem and a corollary which apply to efficient presentations, and a more general extension for other presentations. These are constructive and work very well in practice.

Let G be a simple group or a stem extension of a simple group. Then certain relator amalgamations applied to presentations of G give presentations for stem extensions of G. If a presentation of G includes proper power relators whose base words generate the underlying free group and if combining two of these power relators gives a perfect group then we get a stem extension. More general combination of relators may lead to other preimages.

As the groups of interest become larger, presentation enumeration becomes less effective for finding relevant presentations so most of our results come via generating sets. Asymptotically, almost all pairs of elements of finite simple groups generate the group. Moreover, even for one particular generating pair, there are arbitrarily many different presentations. Our approach enables us to navigate through some of these in structured ways and has proved very effective for finding short presentations for groups which are not too large.

Our general aim is to find concise presentations and possibly longer presentations which are good for computation. We find it useful to classify presentations according to generating pairs, presentation kernels and presentation variants.

It is straightforward in principle to find all generating pairs for our groups. A problem is that, as the groups get larger, the number of generating pairs grows as the square of the group order. Taking the automorphisms of the group into account we reduce the number of generating pairs that we need consider to about the group order divided by a small constant. This is still challengingly large.

For $q \leq 5$ we compute complete sets of generating pairs, as in [14], but this is not well suited to larger groups. We reduce the number that we examine in various ways. In practice for our groups we use conjugacy class representatives to avoid some duplication. Given a pair of classes we select a representative of the larger class and test all members of the smaller class. Sometimes we just take random generating pairs.

3. Interesting presentations

The study of unitary groups and their covering groups has a long history. We intend to describe this more thoroughly in another paper, where we also detail the computations which lead to the results in this section. In Table 2, we highlight a selection of the more interesting presentations and some of their properties for the smaller groups, and do the same for the larger groups in Table 3.

Table 2
Selected presentations of the smaller groups.

Relators, where $u = ab, v = aB$	Group Name	Gen Orders	Total Cosets	Notes
$a^4, aababAbb, aaBababb$	$U_3(2)$	(4,4)	73	METG
$aababAbb, aabbABAb$	$SU_3(2)$	(4,12)	229	METG
$a^3bAbAb, a^2b^2AB^3Ab^2$	$U_3(3)$	(4,12)	198076	MET
$a^2ba^2B^2, a^3bABABAba^3B$	$U_3(3)$	(8,12)	8878	EG
$a^4bAbAb, a^3BAb^2ABA^2B^2$	$U_3(4)$	(5,13)	1557671	MET
$a^2babAbab, a^6b^2AbAb^2$	$U_3(4)$	(15,15)	3953806	ME
$a^2ba^2B^2, ababAB^5AbAb^2$	$U_3(4)$	(10,15)	3939040	ME
$a^2, b^3, u^7v^2u^4vuv^2u^2v^2$	$U_3(4)$	(2,3)	103904	MG
$a^2b^3, (V^7)^Av^2u^4vV^Av^2u^2v^2$	$U_3(4)$	(2,3)	485506	ME
$a^5, a^2bABabaBAb, ab^2Ab^2ab^2AB^2$	$U_3(5)$	(5,4)	1240589	ET
$a^5b^4, (ab)^2AbbaBabABB, (aab)^2Ababb(ab)^2$	$U_3(5)$	(5,4)	186408	EG
$a^4, (abb)^2aB^3, a^2b(aB)^3aba^2B^2$	$SU_3(5)$	(4,5)	19731726	T
$a^3baBAAbaB, a^5b^7, (a^3B)^2BabaBB$	$SU_3(5)$	(15,21)	468456	G

All minimal generating sets for $U_3(2)$ are equivalent modulo automorphisms of the group and the underlying free group. This means that $U_3(2)$ is efficient on all generating pairs. We give a canonical (as in [15]) representative of an efficient 2-generator $(4, 4)$-presentation with shortest length, 20.

There are 36 presentation kernels for $SU_3(2)$ and all have efficient presentations. We give a canonical representative of an efficient, $(4, 12)$-presentation with shortest length, 16. Adding the relator b^4 we get an another shortest length presentation for $U_3(2)$.

The group $U_3(3)$ is discussed in [14] and [3]. Four efficient presentations are listed in [3], including the $(4, 12)$-generated one of Table 2. It has length 19 but is not CE-good. However the length 21, $(8, 12)$-presentation in Table 2 is. The online Atlas presentation is $(2, 6)$-generated, has 5 relators and length 67. The diversity of presentations for this relatively small group is perhaps surprising. There are 2784 presentation kernels on generating sequences.

In [3] five new, efficient presentations for $U_3(4)$ were given. The shortest efficient presentations have length 22, one of which is listed there. There are 3 presentation kernels which have canonical representatives with that length and we list those in Table 2.

By enumerating one-relator quotients of the modular group $\{a^2, b^3\}$, we find that $U_3(4)$ has length 47 and longer presentations. We tabulate a canonical (as in [7]) representative which is CE-good. By examining all shorter, perfect, one-relator quotients, we conclude this is shortest possible. It is a variant of the 5-relator, length 87 presentation in the online Atlas. Theorem 4.1 of [3] gives us many ways to build efficient presentations with the same length from it. For example, we can replace the power relators by their product a^2b^3 and change any of the eight occurrences of a to A in the other relator. We give one instance in Table 2, which is much worse for coset enumeration. There are 2 $(2,3)$-presentation kernels for $U_3(4)$. For the other, we have found length 59, but not shorter, presentations as one-relator quotients.

The group $U_3(5)$ is discussed in detail in [5], which includes a length 28 presentation, which is not CE-good. We also tabulate an efficient, length 38 presentation which is CE-good. The online Atlas presentation has 5 relators and length 50. Relator amalgamation of two pairs of powers in it leads to various other efficient presentations.

No efficient presentation for $SU_3(5)$ is known. This group remains the smallest possible counterexample to the conjecture that the covering groups of all finite simple groups are efficient. Previously found shortest 3-relator presentations have length 30. We have now found one with length 29 which we tabulate. We also give a length 37, CE-good presentation which is a canonical form of a presentation from [4].

Since our presentations for the larger groups are longer, Table 3 is structured differently. Presentations precede group properties on separate lines.

The group $U_3(7)$ was not previously known to be efficient. From random generating sets we found various efficient presentations. From these, `automac` gave shorter efficient presentations with length 24. We tabulate a canonical representative. Since we have such short efficient presentations, we have not analysed other presentations in detail. We did find various reasonably short, 3-relator presentations which are CE-good and tabulate one of them. The online Atlas $(2,3)$-presentation has 6 relators, length 205, but is poor for coset enumeration, using 53248855 total cosets over the trivial subgroup. Our methods give a 5-relator variant $\{a^2, b^3, (u^4v^4)^3, (u^3v^3)^3uv^2uv^3, u^3v^2u^3v^{11}u^6v^2u^2vu^2v^3u^3v^2\}$ which is shorter (length 183) and better for coset enumeration, using 14592554 total cosets.

Two efficient presentations for $U_3(8)$ produced by relator deletion from presentations based on random generating pairs appear in [5]. Applying `automac` to the shorter one, we get the tabulated, length 36 presentation which is almost CE-good.

For $U_3(8)$ there are 2916 $(2,3)$-generating pairs with 2 presentation kernels. We have identified the presentation kernel for standard generators of the online Atlas. We find a 5-relator, length 143 presentation, which is shorter than that in the online Atlas (7 relators, length 203) and is very good for coset enumeration, namely

$$\{a^2, b^3, u^{19}, uv^3uv^3u^2vu^3vu^2v^3, uvuvuv^2uv^4uvuv^4uv^2uvuvuv^3\}$$

which uses 5747911 total cosets over the trivial subgroup and is a practical presentation for these generators. From this we can build a length 127 presentation, replacing a^2 and b^3 by a^2b^3, giving an efficient presentation for $U_3(8)$ on the Atlas standard generators, but it is very difficult for coset enumeration.

Table 3
Selected presentations of the larger groups.

Group Name	Group Order	No of Rels	Pres Length	Gen Orders	Total Cosets	Notes
$aabaabaaB, abAbaBABabAbABB$						
$U_3(7)$	5663616	2	24	(16,16)	20018866	ET
$a^4, aabaBabaaBB, abAbABBAbbabbbABaBB$						
$U_3(7)$	5663616	3	34	(4,43)	7178305	G
$(abaB)^2, a^2ba^2BA^2B, a^3B^2abAb^2A^2b^2AbaB^2$						
$U_3(8)$	5515776	3	36	(21,21)	11300294	ET
$uvUvuV, u(uv)^5vu^2v^5u^4v^3, uv(uv^4)^2uvuv^3uv^4uv^3$						
$U_3(8)$	5515776	3	59	(21,21)	8164048	EG
$a^6bAbAb, ababbabbabaBB, aaabABabAABabaB$						
$SU_3(8)$	16547328	3	39	(7,63)	24855963	TG
$aaabABABAb, abaBAbABBAbAB, aaaBabAbABAAbAB$						
$U_3(9)$	42573600	3	38	(8,40)	51230621	TG
$a^2, b^3, uvuv^7uvuvu^7v, u^2v^3u^2v^3u^2vuvu^2vu^2vuv$						
$U_3(9)$	42573600	4	95	(2,3)	43365169	G
$a^2, b^3, (u^3v^3)^3, uv^2u^3v^2uv^4u^2v^4uv^2u^3v^2$						
$U_3(9)$	42573600	4	95	(2,3)	48669618	G
$a^2, b^3, (u^2v^2)^3, uv^{10}uv^7u^3vu^3v^7$						
$U_3(9)$	42573600	4	95	(2,3)	49089008	G
$a^2, b^3, (u^3v^3)^3, u^2v^6u^2v^6u^2v^4u^2vu^2v^4$						
$U_3(9)$	42573600	4	103	(2,3)	60405092	G
$uvUvuV, uvuv^7uvuvu^7v, u^2v^3u^2v^3u^2vuvu^2vu^2vuv$						
$U_3(9)$	42573600	3	51	(80,80)	47134355	G
$a^3, (abAbb)^2abABB, abAbabbabAbabbABaBAbb$						
$U_3(9)$	42573600	3	39	(3,4)	70214845	G
$a^{10}, b^{10}, Aba^3BA^3b^2, a^2B^3Ab^3aB, aBaBaBABAbAbAb, (aB)^{10}$						
$U_3(11)$	70915680	6	76	(10,10)	73211636	G
$a^2, b^3, (uv^2)^5, u^3v^2uv^3uv^4uv^5u^2vuvu^2v^4u^2v$						
$U_3(11)$	70915680	4	103	(2,3)	1141445437	
$uvUvuV, (uv^2)^5, u^3v^2uv^3uv^4uv^5u^2vuvu^2v^4u^2v$						
$U_3(11)$	70915680	3	55	(37,37)	1676694061	E
$A^3b(ab)^2, abAbaBaBBaB, abABabbaBAbABabbaBAbaBBBB$						
$U_3(11)$	70915680	3	44	(4,20)	86351677	ETG
$a^{10}, a^3bA^3BaBB, aaB^3Ab^3aB, ababAbAbABaBaB$						
$SU_3(11)$	212747040	4	46	(10,30)	236468860	TG
$a^3bAb, a^2baB^2ABAbab^3(ab^2)^2(AB)^2B, a^3B^3A(B^2a^2)^2B^3AB^2a^2B^3(a^2B^2)^4$						
$SU_3(11)$	212747040	3	73	(4,111)	298839543	G

For the other $(2,3)$-presentation kernel we obtain analogous results. They are much better for coset enumeration. We can derive a reasonably short (length 59), efficient presentation on u and v which is CE-good and we tabulate it.

By investigating random generating pairs for $U_3(8)$ we found various 4-relator presentations which include two or three power relators. Then amalgamating power relators we found the tabulated instance of a shortest one for $SU_3(8)$.

No 2-generator presentation for $U_3(9)$ had been written down. No efficient presentation is known, but we have found presentations with one extra relator. From random generating sets we obtained presentations with 3, 4 and 5 relators including one 3-relator presentation which is $(40, 10)$-generated and has length 43. Applying `automac` we found the CE-good, $(8, 40)$-generated presentation with length 38 that we tabulate.

For $U_3(9)$ there are 4536 $(2, 3)$-generating pairs with 14 presentation kernels. They lead to four presentations with 4 relators which we tabulate. All of these 4-relator presentations are CE-good. By replacing a^2 and b^3 by their product a^2b^3 we obtain four different 3-relator ones for $U_3(9)$ which also enumerate well. Then moving to generators that correspond to u and v (replacing a^2, b^3 by $(vUv)^2$, $(Uv)^3$) we get much shorter presentations which enumerate slightly better. Amalgamation of $(vUv)^2$ and $(Uv)^3$ to $uvUvuV$ for these four presentations gives a perfect group in only one case, namely the first, which we tabulate.

We also sampled $(3, 4)$-generating sets, giving more 4-relator presentations suitable for relator amalgamation. We tabulate a shortest one found from these.

No 2-generator presentation for $U_3(11)$ had been written down until recently. Reasonably short presentations for this group can be constructed because it is a section of a group which has a short presentation.

Using presentation enumeration [7] we investigated groups G presented as quotients of the modular group $\{a^2, b^3\}$ by adding one extra relator $w(a, b)$. For w with length up to 36 there are, up to equivalence, 8596 different presentations, and we found out enough to conclude that none of them defines a unitary group.

We did, however, find a finite quotient G with order 3829446720. This group has a length 36 extra relator $u^5vu^2v^2uv^5uv$ (where $u = ab$ and $v = aB$). We observed that $|G|$ is a multiple of the order of $U_3(11)$ and showed that G has $U_3(11)$ as a section. This gives many ways of producing short presentations for $U_3(11)$.

We can obtain a presentation for the index 2 subgroup $H = \langle y, z\rangle$ (where $y = b$ and $z = b^a$) in G by Reidemeister-Schreier rewriting. Now H maps onto $U_3(11)$, and adding the relator $(yz)^{10}$ to the presentation for H gives a presentation for a group which has $U_3(11)$ as an index 3 subgroup. Another rewrite gives the first presentation for $U_3(11)$ in Table 3. Each of the power relators is individually redundant. Deleting any one gives us a 5-relator presentation whose coset enumeration behaviour is only a little worse.

Other ways of looking at the subgroup lattice of G lead to various other short presentations for $U_3(11)$ and for $SU_3(11)$, all of which enumerate quite nicely. Those presentations provide further opportunities for investigation.

For $U_3(11)$ there are 8640 $(2, 3)$-generating pairs with 10 presentation kernels. We can reduce a presentation for one of them to 4 relators, which gives our second presentation in Table 3. It is quite hard for coset enumeration.

Now amalgamating a^2 and b^3 to a^2b^3 gives a perfect group so it is a stem extension. The corresponding enumeration over $\langle uv^2\rangle$ is about ten times harder than for the presentation with unamalgamated relators, and gives the same index. So this 3-relator presentation with length 103 defines $U_3(11)$ and was our first efficient presentation for it. The alternative amalgamation to A^2b^3 also gives a perfect group, again $U_3(11)$, but a better variant for coset enumeration, using about half as many cosets. Instead, moving to a $(37, 37)$-presentation on u and v, replacing a^2 by $(vUv)^2$ and b^3 by $(Uv)^3$, we obtain a much shorter 4-relator presentation. Then, amalgamating the power relators, we obtain a third efficient presentation which we tabulate. It is shorter, length 55, and better for coset enumeration but still quite difficult.

We can obtain a CE-good, efficient presentation. We sampled $(3, 4)$-generating sets. We applied `automac` to their presentations and obtained as shortest for $U_3(11)$ a $(4, 20)$-generated one. Amalgamating the power relators in that, we get a shortest efficient presentation so far discovered, which is CE-good and we tabulate it.

The shortest presentations we have for $SU_3(11)$ are derived from our first presentation for $U_3(11)$. We tabulate a canonical example with 4 relators and length 46. We are able to find 3-relator presentations for $SU_3(11)$ by looking at the 10944 $(2, 4)$-generating pairs which have 38 presentation kernels. From one of these we obtain a $(4, 111)$-presentation for $SU_3(11)$ which is CE-good and tabulated.

4. Concluding remarks

We have short presentations on generating pairs for the 3-dimensional unitary groups $U_3(q)$ and $SU_3(q)$ for q up to 11. We have not needed to go beyond length 46 for our shortest presentation for any of them. The efficiency question remains unresolved for the perfect proper covers, $SU_3(5)$, $SU_3(8)$ and $SU_3(11)$, and for one of the simple groups, $U_3(9)$, all of which have trivial multiplier. Efficiency with 2 generators means 2 relators, but the closest we have found are 3-relator presentations.

There are still many unresolved questions for unitary groups. Are they all efficient? If so, are they efficient on every generating set? What are their shortest presentations? What are the shortest presentations on generating pairs? What are the shortest efficient presentations? Which unitary groups are one-relator quotients of the modular group? We have answered these questions in some cases for the groups we study.

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