Automorphism Groups of Certain Non-Quasiprimitive Almost Simple Graphs

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Abstract
A graph $\Gamma$ is $G$-quasiprimitive if $G$ is a group of automorphisms of $\Gamma$ such that each nontrivial normal subgroup of $G$ acts transitively on the vertices of $\Gamma$. We consider $\Gamma$ a finite, connected $S$-locally-primitive graph with $S$ a nonabelian simple group and give a set of conditions under which we may guarantee that either the full automorphism group of $\Gamma$ is not quasiprimitive or there is a non-quasiprimitive subgroup $Y$ of $\text{Aut}\Gamma$ such that $C.G$ is maximal in $Y$, where $C$ is the centralizer of $S$ in $\text{Aut}\Gamma$ and $G$ is an almost simple group with socle $S$. As an application of this result we show that, in certain circumstances, $\text{Aut}\Gamma = Z_p.G$, where $p$ is a prime and $G = \text{Aut}\Gamma \cap \text{Aut}(S)$.

1 Introduction
A permutation group on a set $\Omega$ is said to be quasiprimitive if each of its nontrivial normal subgroups is transitive on $\Omega$. By the structure theorem in [18, Section 2] quasiprimitive permutation groups may be divided into 8 disjoint types ($HA$, $HS$, $HC$, $AS$, $SD$, $CD$, $TW$, $PA$) in a way which is helpful for applications to graph theory (see the description in [3]). The main type considered in this paper is $AS$: a quasiprimitive group $G$ is said to be of type $AS$ if $S \leq G \leq \text{Aut}(S)$, for some nonabelian simple group $S$.

Let $\Gamma$ be a finite simple undirected graph with vertex set $V\Gamma$ and edge set $E\Gamma$ and let $G$ be a group of automorphisms of $\Gamma$. The graph $\Gamma$ is said to be $G$-quasiprimitive if and only if $G$ is quasiprimitive on $V\Gamma$. In particular, an $\text{Aut}\Gamma$-quasiprimitive graph is said to be quasiprimitive. The graph $\Gamma$ is said to be $G$-locally-primitive if $G$ is transitive on $V\Gamma$ and the stabilizer $G_\alpha$ is primitive on

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\( \Gamma(\alpha) \), where \( \Gamma(\alpha) \) denotes the set of all vertices adjacent to \( \alpha \). The graph \( \Gamma \) is said to be \( (G, 2) \)-arc transitive if \( G \) is transitive on the 2-arcs of \( \Gamma \).

Praeger in [19] raises the questions: (1) for a \( G \)-quasiprimitive graph \( \Gamma \), under what conditions can we be certain that \( \text{Aut}\Gamma \) is quasiprimitive on \( V\Gamma \)? (2) If \( \text{Aut}\Gamma \) and \( G \) are quasiprimitive with the same quasiprimitive type, is it possible that \( \text{soc}(G) \neq \text{soc}(\text{Aut}\Gamma) \), and if so what are the possibilities for these socles?

We look first at question (1). It is shown in [7] that, for a connected \( G \)-quasiprimitive graph \( \Gamma \) with \( G \) of type \( AS \), if \( \Gamma \) is \( G \)-locally-primitive then either \( \Gamma \) is quasiprimitive or \( \text{Aut}\Gamma \) has a restricted structure. Baddeley in [2, Section 6] constructs the first example of connected non-quasiprimitive graph \( \Gamma \) which is \( (G, 2) \)-arc transitive, for some quasiprimitive group \( G \) of type \( TW \), and he comments there that such graphs seem difficult to construct. An infinite family of \( (L_2(q), 2) \)-arc transitive graphs with valency 4 is constructed in [12], and Li [14] proves that the full automorphism groups are isomorphic to \( Z_2 \times L_2(q) \), which gives the first infinite family of non-quasiprimitive graphs \( \Gamma \) such that \( L_2(q) \) acts quasiprimitively on \( V\Gamma \). Recently an infinite family of \( U_3(q) \)-quasiprimitive transitive graphs of valency 9 has been constructed by the authors of this paper in [8]; their full automorphism groups are isomorphic to \( Z_3 \times G \) with \( Z_3 = C_{\text{Aut}}(U_3(q)) \) and \( U_3(q) \leq G \leq \text{Aut}(U_3(q)) \). From this we obtain a new infinite family of non-quasiprimitive graphs \( \Gamma \) admitting a quasiprimitive group of type \( AS \) acting transitively on the vertices and the 2-arcs of \( \Gamma \). Observing all non-quasiprimitive examples with \( G \) quasiprimitive given above, we find that \( \text{Aut}\Gamma = C \cdot G \) with \( C \) some small nontrivial cyclic group, which motivates us to suggest a set of conditions under which we may guarantee that either \( \Gamma \) is non-quasiprimitive or \( \text{Aut}\Gamma \) has a restricted structure. In this paper we concentrate on \( G \)-quasiprimitive graphs for \( G \) of type \( AS \). The set of conditions we employ is the following

**Condition \( \mathcal{P} \).** Let \( \Gamma \) be a connected, \( S \)-locally-primitive graph with \( S \) a nonabelian simple group. Set \( G = \text{Aut}(S) \cap \text{Aut}\Gamma \) and \( C = C_{\text{Aut}}(S) \). The graph \( \Gamma \) and \( G \) are said to satisfy Condition \( \mathcal{P} \) if \( C \neq 1 \) and \( \text{Aut}\Gamma \) has no quasiprimitive subgroup of type \( AS \) containing \( C \cdot G \) as its maximal subgroup.

**Remarks on Condition \( \mathcal{P} \)**

(a) Using [7, Theorem 1.2] we shall prove that, for a connected \( S \)-locally-primitive graph \( \Gamma \) with \( S \) a nonabelian simple group with \( \Gamma \) and \( G \) satisfying Condition \( \mathcal{P} \), then any quasiprimitive overgroup \( Y \) of \( G \) in \( \text{Aut}\Gamma \) must be of type \( AS \) (see Theorem 1.1). From this we conclude that Condition \( \mathcal{P} \) does guarantee that either \( \Gamma \) is non-quasiprimitive or \( \text{Aut}\Gamma \) has a non-quasiprimitive subgroup \( Y \) such that \( C \cdot G \) is maximal in \( Y \).

(b) For an arbitrary group \( C \), it is in general quite difficult to verify that \( \Gamma \) and \( G \) satisfy Condition \( \mathcal{P} \). However, if \( C \) is a nilpotent group and if \( p \) is a prime divisor of \( |C| \), then \( C \cdot G \) is a maximal \( p \)-local subgroup of \( Y \) whenever there exists such a quasiprimitive group \( Y \) of type \( AS \) in \( \text{Aut}\Gamma \). Note that all maximal \( p \)-local subgroups of almost simple groups have been determined (see, for example, [1, 5, 13, 16]). So we can find out which \( \Gamma \) and \( G \) satisfy Condition \( \mathcal{P} \) using a case-by-case check.
The main results of this paper are the following.

**Theorem 1.1** Suppose that $\Gamma$ and $G$ satisfy Condition $\mathcal{P}$. Then either $\Gamma$ is a non-quasiprimitive graph; or $\text{Aut}\Gamma$ is a quasiprimitive group of type $AS$ and $\text{Aut}\Gamma$ contains a non-quasiprimitive subgroup $Y$ such that $C.G$ is maximal in $Y$. Further, for any intransitive minimal normal subgroup $N$ of $Y$, then: $N$ centralizes $S$; or $N \cap C = 1$ and $N = Z^n_p$ for some prime $p$; and $S = S(q)$ is a simple group of Lie type over a field of order $q = p^e$, and further, $N$ is the unique intransitive minimal normal subgroup of $\text{Aut}\Gamma$ not centralized by $S$.

For $\Gamma$ and $G$ given as in Theorem 1.1, let $\Gamma^*$ denote the quotient graph modulo the $C$-orbits on $V\Gamma$ (obtained by taking $C$-orbits as vertices and joining two $C$-orbits by an edge if there is at least one edge in $\Gamma$ joining a point in the first $C$-orbit to a point in the second one). By [17], $\Gamma$ is a cover of $\Gamma^*$. As an application of Theorem 1.1 we have the following.

**Theorem 1.2** Let $\Gamma$ and $G$ satisfy Condition $\mathcal{P}$. Suppose that $C$ is a cyclic group with prime order and that $\text{Aut}\Gamma$ has no a subgroup $N:S$ given as in Table 1. If $\text{Aut}\Gamma^*$ is quasiprimitive of type $AS$ with socle $S$, then $\Gamma$ is non-quasiprimitive and $\text{Aut}\Gamma = C:G$.

**Remark on Theorem 1.2**

For each graph given as in [8, 14], we know that $C = Z_3$ or $Z_2$ and that $\text{Aut}\Gamma$ has no subgroup $N.S$ given as in Table 1. It is also easy to prove that $S \leq \text{Aut}\Gamma^* \leq \text{Aut}(S)$ (up to isomorphism). By Theorem 1.2, to show $\text{Aut}\Gamma = C.G$, it remains only to verify that $\Gamma$ and $G$ satisfy Condition $\mathcal{P}$, which can be completed by using the method mentioned in Remark (b) on Condition $\mathcal{P}$. Indeed, this is not difficult to check for the graphs given in [8, 14].

Theorems 1.1 and 1.2 are proved in the next section. In the final section we consider question (2) raised at the beginning of this paper and give some examples of $G$-quasiprimitive graphs such that $\text{Aut}\Gamma$ and $G$ have same quasiprimitive type but $\text{soc}(\text{Aut}\Gamma) \neq \text{soc}(G)$.

<table>
<thead>
<tr>
<th>$S$</th>
<th>values for $n/e$</th>
<th>$S$</th>
<th>values for $n/e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_l(q), l = 3$ or $4$</td>
<td>$[8, 9]$ or $[8, 16]$</td>
<td>$E_7(q)$</td>
<td>$[56, 63]$</td>
</tr>
<tr>
<td>$C_l(q), l = 3$ or $4$</td>
<td>$[8, 9]$ or $[9, 16]$</td>
<td>$A^\pm_l(q)$, $l &gt; 1$</td>
<td>$[l + 1, l(l + 1)/2]$</td>
</tr>
<tr>
<td>$D^+_l(q), l = 4$ or $5$</td>
<td>$[8, 12]$ or $[16, 20]$</td>
<td>$B_l(q)$, $l &gt; 4$</td>
<td>$[2l + 1, l^2]$</td>
</tr>
<tr>
<td>$3D_4(q), q$ odd</td>
<td>$12$</td>
<td>$C_l(q), l \neq 1, 3, 4$</td>
<td>$[2l, l^2]$</td>
</tr>
<tr>
<td>$E_{6^+}^l(q)$</td>
<td>$[27, 36]$</td>
<td>$D^+_l(q), l \neq 1, 2, 4, 5$</td>
<td>$[2l, l(l - 1)]$</td>
</tr>
</tbody>
</table>

Table 1
where $[i, j]$ stands $i \leq n/e \leq j$, for distinct integers $i$ and $j$ with $i < j$. 

For $\Gamma$ and $G$ given as in Theorem 1.1, let $\Gamma^*$ denote the quotient graph modulo the $C$-orbits on $V\Gamma$ (obtained by taking $C$-orbits as vertices and joining two $C$-orbits by an edge if there is at least one edge in $\Gamma$ joining a point in the first $C$-orbit to a point in the second one). By [17], $\Gamma$ is a cover of $\Gamma^*$. As an application of Theorem 1.1 we have the following.
2 Proof of Theorems 1.1 and 1.2

The first lemma is used in the proof of Theorem 1.1. Its proof follows immediately from [17] and the connectivity of \( \Gamma \).

**Lemma 2.1** Let \( \Gamma \) be a connected, nonbipartite, \( Y \)-locally-primitive graph. Then each intransitive normal subgroup of \( Y \) is semiregular.

Suppose now that \( G \) is an almost simple group with socle \( S \) and let \( G \) and \( \Gamma \) be given as in Theorem 1.1. Write \( X := \text{Aut}\Gamma \) and \( C := C_X(S) \). Since \( S \) is locally primitive on \( VT \), \( S \) is transitive on the arcs of \( \Gamma \), and so \( S \) is not regular on \( VT \).

It follows that \( C \) is not transitive on \( VT \) (see, for example, [9, Proposition 2.4] or [21]).

**Proof of Theorem 1.1** First we show that, for each overgroup \( K \) of \( S \) in \( X \), if \( K \) is quasiprimitive then \( K \) is of type \( AS \). Suppose to the contrary that \( K \) is not of type \( AS \). By [7, Theorem 1.2], either \( \Gamma = K_3 \) with \( S = \text{PSL}(2,7) \) and \( K = \text{AGL}(3,2) \); or \( K \) is of type \( PA \) with \( \text{soc}(K) = S_1 \times S_2 \cong S \times S \) and \( S \) is a diagonal subgroup of \( \text{soc}(K) \). In the former case, by [6], \( S_\alpha = Z_7:Z_3 \) and \( S_\alpha \) is a maximal subgroup, and hence \( S \) acts primitively on \( VT \). It follows from [21] that \( C = 1 \), which contradicts Condition \( P \). In the latter case, since \( S < \text{soc}(K), \Gamma \) is also \( \text{soc}(K) \)-locally-primitive. Since \( S_i < \text{soc}(K) \) and \( S_i \cong S \), for \( i = 1,2 \), the both \( S_1 \) and \( S_2 \) are transitive by Lemma 2.1. It follows that \( S_1 \) and \( S_2 \) are regular, which is impossible since \( |S_1| = |S| > |VT| \). So \( K \) is of type \( AS \). In particular, take \( K = X \), then \( X \) is of type \( AS \).

Suppose now that \( X \) is quasiprimitive. Since \( X \) is of type \( AS \), by Condition \( P \), \( G.C \) is not maximal in \( X \), and hence there is a proper subgroup \( Y \) of \( X \) containing \( G.C \) such that \( G.C \) is maximal in \( Y \). If \( Y \) is quasiprimitive then \( Y \) is of type \( AS \), which contradicts Condition \( P \). Thus \( Y \) is not quasiprimitive.

Let \( N \) be an intransitive minimal normal subgroup of \( Y \) with \( N \not\leq C \). Since \( \Gamma \) is also a \( Y \)-locally-primitive graph, by Lemma 2.1, \( N \) is semiregular on \( VT \). If \( N < C.G \) then \( CN/C \) is isomorphic to a normal subgroup of \( G \), which implies that either \( N \leq C \) or \( S \) isomorphic to a subgroup of \( CN/C \). The former case contradicts \( N \not\leq C \). While in the latter case we conclude that \( |N| \) is divisible by \( |S| \), which contradicts the fact that \( N \) is semiregular. Thus \( N \not\leq C.G \) and so \( Y = \langle N, C.G \rangle \) (since \( C.G \) is maximal in \( Y \)). Then arguing as in the proof of [7, Theorem 1.1] we conclude that \( N \) is an elementary abelian \( p \)-group, for some prime \( p \). If \( N \cap C \neq 1 \), then \( N \cap C \) is normalized by \( N \) and \( C.G \), respectively. Thus \( N \cap C \) is a nontrivial normal subgroup of \( Y \). Since \( N \cap C \leq C \) and \( N \) is a minimal normal subgroup of \( Y \), \( N \cap C = N \) and hence \( N \not\leq C \), which is not the case. So \( N \cap C = 1 \). It follows that \( S \) acts faithfully by conjugation on \( N \). Thus \( S \) has a faithful projective \( p \)-modular representation of degree \( n \). A similar argument as in the proof of [10, Theorem 1.1] shows that \( S \) and \( n/e \) are given as in Table 1. For example, we look at the case where \( S = A_l(q) \) with \( q = p_1^t \) for some prime \( p_1 \). If \( p_1 \neq p \) then \( n/e \geq (q-1)/(2q-1) \) by [13, Table 5.3.A], and hence
\[
|N| \geq q^{t(q-1)/(2q-1)} \geq |A_l(q)| > |VT|.
\]
which is impossible. So \( p_1 = p \). Now let \( R_p(S) \) denote the minimal dimension of a faithful, irreducible, projective \( K \)-module, where \( K \) is an algebraically closed field of characteristic \( p \). By [13, Table 5.4.C], \( R_p(S) = l + 1 \), which implies that \( n/e \geq l + 1 \). On the other hand, since a Sylow \( p \)-subgroup of \( S \) has order \( q^{\ell(l+1)/2} \) and since \( |S| \) is divisible by \( |N| \), \( n/e \leq l(l+1)/2 \). So \( n/e \) lies in \([l+1,l(l+1)/2]\). A similar argument deals with other nonabelian simple groups \( S \).

Finally, for \( S \) in the first column of Table 1 or \( E_7(q) \), if there is another intransitive minimal normal subgroup \( K \) not centralized by \( S \), then a similar argument as above implies that \( K \cong Z_p^n \) with \( m/e \) in the second column of Table 1 or [56, 63]. Set \( W = NK \). Now \( W \) is a normal \( p \)-subgroup of \( X \) with \( |W| = p^{a+m} \). Since \( p^{a+m} \) is greater than the \( p \)-part of \( |S| \), \( W \) must be transitive on \( VT \) by Lemma 2.1. Recall that \( S \) is one of the following groups: \( B_l(q) \) or \( G_l(q) \) with \( l \in \{3, 4\} \), \( D_l^\pm(q) \) with \( l = 4 \) or \( 5 \), \( 3D_4(q) \) with \( q \) odd or \( E_7(q) \). From [11, Corollary 2] it follows that \( S = \text{PSp}(4,3) \) and \( |VT| = 27 \), and so \( X \leq \text{AGL}(3,3) \). However it is trivial to see that \( \text{PSp}(4,3) \) is not isomorphic to a subgroup of \( \text{AGL}(3,3) \), which is a contradiction. So \( N \) is the unique intransitive minimal normal subgroup of \( X \) not centralized by \( S \).

\[ \square \]

**Proof of Theorem 1.2** We claim first that \( \text{Aut}\Gamma \) is not quasiprimitive. If this is not the case, by Theorem 1.1, \( \text{Aut}\Gamma \) contains a non-quasiprimitive subgroup \( Y \) such that \( C \cdot G \) is maximal in \( Y \). Let \( N \) be an intransitive minimal normal subgroup \( Y \). If \( N \not\leq C \), by Theorem 1.1, we conclude that \( N:S \) would be in Table 1, which is not the case. So \( N \leq C \) and hence \( N = C = Z_{p_1} \). Since \( G \cong (C \cdot G)/C < Y/C \) and \( Y/C \) is isomorphic to a subgroup of \( \text{Aut}\Gamma^* \), \( S \leq Y/C \leq \text{Aut}(S) \) (up to isomorphism). It follows that \( Y \cong C \cdot G_1 \), for some group \( G_1 \) with \( S \leq G_1 \leq \text{Aut}(S) \). On the other hand, from the definition of \( G \) we know that \( G \) is the maximal group among the groups \( H \) with the properties that \( S \leq H \leq \text{Aut}(S) \) and \( C \cdot H \leq \text{Aut}\Gamma \), which implies that \( G_1 \cong G \). Thus \( Y = C \cdot G \), which is a contradiction. So \( \text{Aut}\Gamma \) is non-quasiprimitive.

Let \( N \) be an intransitive minimal normal subgroup of \( \text{Aut}\Gamma \). If \( N \not\leq C \) then \( S \) acts nontrivially by conjugation on \( N \). Now \( N \) is semiregular on \( VT \) by Lemma 2.1. Then arguing as in the proof of Theorem 1.1, we conclude that \( N = Z_{p_1}^n \), for some prime \( p \) and integer \( n > 1 \), and \( N:S \) lies in Table 1, which is a contradiction. So \( N \leq C \), and hence \( N = C \). Now \( C \triangleleft \text{Aut}\Gamma \) and \( \text{Aut}\Gamma/C \) is a subgroup of \( \text{Aut}\Gamma^* \). Thus \( \text{Aut}\Gamma \cong C \cdot G_1^* \), for some subgroup \( G_1^* \) of \( \text{Aut}\Gamma^* \) with \( S \leq G_1^* \leq \text{Aut}(S) \) (up to isomorphism). From the definition of \( G \) it follows that \( G_1^* \cong G \) and hence \( \text{Aut}\Gamma = C \cdot G \).

\[ \square \]

### 3 Some examples of quasiprimitive graphs

In this section we always assume that \( G \) is an almost simple group and \( \Gamma \) is a connected \( G \)-arc transitive graph with valency \( d\Gamma \). By [20] there exists a 2-element \( g \in G \) with the properties: \( g \notin N_G(H) \), \( g^2 \in H \) and \( \langle H, g \rangle = G \), such that
\[ \Gamma \cong \Gamma^* := \Gamma(G, H, HgH) \] with \( d_\Gamma = |H : H \cap H^g| \), where \( \Gamma^* \) is defined by

\[ V \Gamma^* = \{ Hx \mid x \in G \}, \quad E \Gamma^* = \{ \{ Hx, Hy \} \mid x, y \in G, xy^{-1} \in HgH \}. \tag{1} \]

All connected regular \( G \)-arc transitive graphs considered in this section will be defined in terms of a subgroup \( H \) and a 2-element \( g \) as in (1).

Now we give some examples of \( G \)-quasiprimitive graphs. The first example gives all \( G \)-quasiprimitive graphs with a prime power number of vertices, for \( G \) a nonabelian simple group. The proof of Example 3.1 follows immediately from [11, Corollary 2] and [6].

**Example 3.1** Let \( \Gamma \) be a finite connected graph and suppose that \( G \) is a transitive subgroup of \( \text{Aut}\Gamma \) with \( G \) a nonabelian simple group. If \( |V\Gamma| \) is a prime power, then either \( \Gamma \) is a complete graph or \( G = PSU(4, 2) \) and \( |V\Gamma| = 27 \). Further, \( G \) and \( \text{soc}(\text{Aut}\Gamma) \) are given in Table 2.

| \( G \)          | \( |V\Gamma| \) | \( \text{soc}(\text{Aut}\Gamma) \) | comments on \( G_\alpha \) |
|----------------|----------------|-------------------------------|--------------------------|
| \( A_{p^a} \)   | \( p^a \)      | \( A_{p^a} \)                | \( G_\alpha \cong A_{p^a-1} \) |
| \( \text{PSL}(n, q) \) | \( \frac{q^n-1}{q-1} = p^a \) | \( A_{p^a} \)                | the stabilizer of a line or hyperplane |
| \( \text{PSL}(2, 11) \) | 11            | \( A_{11} \)                | \( G_\alpha \cong A_5 \) |
| \( M_{23} \)    | 23             | \( A_{23} \)                | \( G_\alpha \cong M_{22} \) |
| \( M_{11} \)    | 11             | \( A_{11} \)                | \( G_\alpha \cong M_{10} \) |
| \( \text{PSU}(4, 2) \) | 27            | \( \text{PSU}(4, 2) \)      | \( G_\alpha \cong 2^4.A_5 \) |

**Table 2**

The graphs on the first five lines of Table 2 are all complete, but the graph for \( \text{PSU}(4, 2) \) is not. The graphs on lines two to five provide one infinite family and three particular graphs with the property that \( \text{soc}(G) \neq \text{soc}(\text{Aut}\Gamma) \).

**Example 3.2** Let \( \Gamma \) be a connected \( G \)-arc transitive graph of valency \( d_\Gamma \), where \( (G, |V\Gamma|) \) is one of \( (\text{PGL}(2, 8), 36) \), \( (A_9, 120) \) and \( (M_{11}, 55) \). Then \( \text{Aut}\Gamma \) is an almost simple group. Moreover, the pair \( (G, \text{soc}(\text{Aut}G)) \) is \( (\text{PGL}(2, 8), A_9) \), \( (A_9, \Omega_5^+(2)) \) or \( (M_{11}, A_{11}) \), respectively.

**Proof.** For \( \alpha \in V\Gamma \) write \( H = G_\alpha \). For \( (G, |V\Gamma|) \) given as above, by [6] we have the structure of \( H \) as given in Table 3 (see Column 4). Note that \( H \) is a maximal subgroup of \( G \), for \( G \) and \( G_\alpha \) given as in Table 3. So \( \langle H, g \rangle = G \), for any 2-element \( g \in G \setminus H \), and hence the graph \( \Gamma = \Gamma(G, H, HgH) \) is a connected \( G \)-arc transitive graph with valency \( d_\Gamma = |H : H \cap H^g| \). Look first at \( G = \text{PGL}(2, 8) \) and \( H = R:T \) with \( R \cong Z_7 \) and \( T \cong Z_6 \). Computation shows that \( T \) is self-normalized in \( G \). Thus \( |H \cap H^g| \) is either 2 or 3, and so \( d_\Gamma = 21 \) or 14. Moreover, since a Sylow 2-subgroup of \( G \) is elementary abelian all suitable 2-elements \( g \) have order 2. If \( d_\Gamma = 21 \) then \( H \cap H^g = 2 \). Using the MAGMA program (see Figure 1) we obtain both all connected \( \text{PGL}(2, 8) \)-arc transitive graphs with valency 21 and their
```plaintext
H := PGammaL(2,8);
A := SylowSubgroup(H, 7);
M := Normalizer(H, A);
for x in M do
  if Order(x) eq 3 then
    for y in M do
      if Order(y) eq 2 then
        M6 := sub< H | x, y >;
        if Order(M6) eq 6 then
          B := M6; break;
        end if;
      end if;
    end for;
  end if;
end for;
B3 := SylowSubgroup(B, 3);
B2 := SylowSubgroup(B, 2);
A21 := sub< H | A, B3 >;
N2 := Normalizer(H, B2);

phi, G := CosetAction(H, M);
T := Stabilizer(G, 1);
A21 := phi(A21);
B2 := phi(B2);
NB2 := Normalizer(G, B2);

Grphs := [];
for g in NB2 do
  if Order(g) eq 2 and sub< G | T, g > eq G then
    print "Success with", g;
    found := true;
    Nbs := { 1^x : x in A21 };
    Gr := Graph< Support(G) | <1, Nbs>^G >;
    print Gr;
    print (AutomorphismGroup(Gr));
    print Order(AutomorphismGroup(Gr));
    Append(~Grphs, Gr);
  end if;
end for;
```

Figure 1: MAGMA program for PTL(2,8)-arc transitive graphs of valency 21
full automorphism groups. Using a similar method we obtain Table 3, and the conclusion follows from Columns 4 and 5 of Table 3.

\[\square\]

**Remark on Example 3.2** Recall the definition of 2-closure of a finite permutation group: if $G$ is a finite permutation group on a set $\Omega$ the 2-closure $G^{(2)}$ of $G$ is the largest subgroup of $\text{Sym}(\Omega)$ containing $G$ which has the same orbits as $G$ in the induced action on $\Omega \times \Omega$. For each group $G$ in Example 3.2, Table 3 shows that $\text{Aut}\Gamma \cong G^{(2)}$, and all pairs of $(G, \text{Aut}\Gamma)$ in the example occur in [15, Table 1].

\[
\begin{array}{|c|c|c|c|c|}
\hline
G & d_\Gamma & |V_\Gamma| & G_\alpha & \text{soc}(G) & \text{Aut}\Gamma \\
\hline
\text{PGL}(2, 8) & 14 \text{ or } 21 & 36 & 7:6 & \text{PSL}(2, 8) & S_9 \\
A_9 & 56 \text{ or } 63 & 120 & \text{PSL}(2, 8).2 & A_9 & \Omega^+_8(2) \\
M_{11} & 18 \text{ or } 36 & 55 & 3^2.Q_8.2 & M_{11} & S_{11} \\
\hline
\end{array}
\]

Table 3

**Acknowledgements**

We are grateful to John Cannon for assistance in setting up the MAGMA programs for calculating general arc-transitive graphs. The second author was partially supported by the Australian Research Council.

**References**


