

A Robust Method for Estimating the Fundamental Matrix

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Abstract. In this paper, we propose a robust method to estimate the fundamental matrix in the presence of outliers. The new method uses random minimum subsets as a search engine to find inliers. The fundamental matrix is computed from a minimum subset and subsequently evaluated over the entire data set by means of the same measure, namely minimization of 2D reprojection error. A mixture model of Gaussian and Uniform distributions is used to describe the image errors. An iterative algorithm is developed for estimating the outlier percentage and noise level in the mixture model. Simulation results are provided to illustrate the performance of the proposed method.

1 Introduction

A basic problem in computer vision is recovering 3D construction of a world scene or object from a pair of images. There are many approaches developed to solve this problem, which can be classified into stratified reconstruction methods [1] and direct reconstruction methods [2]. Projective reconstruction is a necessary first step in all these approaches.

The epipolar geometry describes two-view projective geometry. The epipolar geometry can be expressed in terms of the fundamental matrix. The fundamental matrix contains all geometric information necessary for establishing point correspondences between two images, from which projective reconstruction of the scene or object can be inferred. Therefore, projective reconstruction for two views becomes a problem of estimating the fundamental matrix.

The fundamental matrix is estimated using point correspondences between two images. The difficulties in estimating the fundamental matrix lies in the fact that there are often a fair proportion of mismatches in a given set of point correspondences. It is therefore important that the method used for estimating the fundamental matrix should be robust in the presence of mismatches.

There existing several methods for the robust estimation of the fundamental matrix [3]. **M**-Estimators [4, 5] reduce effect of inliers with large noise or outliers by applying weight functions, reducing the problem to a WLS (weighted least-squares) problem. Practically, **M**-Estimators are not so robust to outliers. RANSAC (RANDOM Sample Consensus) technique [2] is a simple and successful method in robust estimation. RANSAC removes the effect of outliers by using random sampling as a search engine for inliers in the data set. Each solution is determined by a random

minimum subset and evaluated against the entire data set for consistency. In detail, RANSAC calculates the number of supporting correspondences for each solution and the one that maximizes the support is chosen. A group of robust methods are developed based on RANSAC. LMedS [6] (Least Median Squares) is similar to RANSAC, except for the way to determine the best solution. LMedS evaluates each solution in terms of the median Symmetric Epipolar Distances of the data set and chooses the one which minimizes this median. The MLESAC [7] (Maximum Likelihood Sample Consensus) is a generalization of RANSAC using the same point selection strategy. MLESAC maximizes a likelihood which is a mixture model of normal distribution (for inliers) and uniform distribution (for outliers) instead of the number of supporting correspondences. The parameter of the model is estimated by expectation maximization (EM). MAPSAC [8] (Maximum A Posteriori Sample Consensus) improves MLESAC by maximizing the posterior estimation of the fundamental matrix and matches. MAPSAC provides new evaluations to check the consistency of each solution with the data set.

In this paper, we will propose two improvements to the robust estimation of the fundamental matrix. First, in RANSAC like methods, the result will depend critically on the choice of a scoring function for evaluating solutions. The scoring function is defined in terms of parameters such as the proportion of inliers γ and the noise level σ corrupting the inliers. It is essential that reasonable estimates of these parameters be provided. In the MLESAC method of [7], γ is estimated iteratively, but σ is estimated only a priori and a posteriori (before and after) the estimation of γ . Since the scoring function depends directly on both γ and σ , we propose in this paper that a better way is to estimate γ and σ together within an iterative algorithm. This would provide a more accurate estimate for both parameters. The accuracy of these parameters has an implication on the reliability of the scoring function. Secondly, in most methods, the fundamental matrix is estimated from a minimum subset using the seven-point [2] or the eight-point algorithm [9], which minimize an algebraic error with no geometric meaning. We propose that a minimization of the 2D reprojection error in computing the fundamental matrix is more appropriate. This is because the fundamental matrix is then evaluated in terms of reprojection errors associated with image points of the entire data set. It is important that a consistent measure is used in the initial determination and subsequent evaluation of the fundamental matrix, as these errors will be used in a scoring function for discriminating inliers and outliers.

This paper is organized as follows. Section 2 briefly introduces robust estimation of the fundamental matrix. In section 3, the proposed robust method is described in detail. In section 4, simulation results are provided to illustrate the performance of the proposed method. In section 5, a conclusion is given.

2 Robust estimation of the fundamental matrix

Consider point correspondences in two images projected from an object. The set of points $\{\mathbf{x}_i\}, i=1, \dots, n$ in the first image and the set of corresponding points $\{\mathbf{x}_i'\}, i=1, \dots, n$ in the second image are related by

$$\mathbf{x}_i^T \mathbf{F} \mathbf{x}_i = 0, \quad i = 1, \dots, n \quad (1)$$

where the 3×3 matrix \mathbf{F} is the fundamental matrix and $\mathbf{x}_i = (x, y, 1)^T$ is in homogeneous form. The fundamental matrix has seven degrees of freedom because it is of rank two and defined only up to scale. Therefore at least seven point correspondences are required for calculating the fundamental matrix.

The fundamental matrix is computed from a set of point correspondences $S = \{(\mathbf{x}_i, \mathbf{x}_i') | i = 1, \dots, n\}$. Points are recognized as corner points by a corner detector and matched as correspondences by a matching algorithm. In practice a set of point correspondences may be corrupted by noise or contain mismatched correspondences. So it is necessary to discriminate the set into inliers and outliers. Inliers refer to correctly matched correspondences and outliers refer to mismatched correspondences.

For a data set with a given proportion γ of inliers, the number of trials N required to give sufficiently high probability p to pick an outlier-free subset consisting of r point correspondences is

$$N = \log(1 - p) / \log(1 - \gamma^r) \quad (2)$$

The general framework of existing robust methods can be summarized as followings:

1. Repeat N times where N is adjusted adaptively using (2) when γ is updated.
 - a. Select a minimal subset S_{sub} from the data set S according to a sample selection strategy.
 - b. Compute \mathbf{F}_j from the subset S_{sub} (e.g. seven-point algorithm, eight-point algorithm)
 - c. Evaluate the consistency of \mathbf{F} with all point correspondences of the data set S .
 - i. Calculate error $\{e_i^2\}$ for each point of the data set S with \mathbf{F}_j .
 - ii. Evaluate \mathbf{F}_j by a scoring function which discriminates the data set into inliers and outliers.
2. Select best solution over a set of solutions $\{\mathbf{F}_j\}$, $j = 1, \dots, N$ according to their scores.
3. Steps may be performed to improve the result of step 2.

Various robust estimation methods (like RANSAC, MLESAC and MAPSAC) differ in one step or another, with step 1(c) being the focus of recent research interest.

3 A new robust method

We propose a robust method which makes two improvements within the above general framework, namely in stage 1(b) and 1(c)(ii). In stage 1(b), existing methods typically use the seven-point or the eight-point algorithm to determine F by minimizing an algebraic error and a different error measure may be used in step 1(c) for evaluating F . For reasons of consistency, we would suggest (see Section 3.1) to use the minimization of the 2D reprojection error both in the determination and the evaluation of F . In stage 1(c)(ii), the evaluation of the scoring function requires the estimation of the γ and σ . We will propose a new iterative algorithm for estimating these parameters (see Section 3.2).

3.1 Determination and evaluation of F by minimization of 2D reprojection error

The 2D reprojection error of all corresponding points in a pair of images is defined as

$$e = \sum_i e_i = \sum_i d(\mathbf{x}_i, \hat{\mathbf{x}}_i)^2 + d(\mathbf{x}_i', \hat{\mathbf{x}}_i')^2 \quad \text{subject to} \quad \hat{\mathbf{x}}_i' \mathbf{F} \hat{\mathbf{x}}_i = 0 \quad \forall i. \quad (3)$$

It can be interpreted geometrically. Each correspondence $\mathbf{x}_i(x_i, y_i, 1)$, $\mathbf{x}_i'(x_i', y_i', 1)$ defines a single point denoted $\mathbf{X}_i(x_i, y_i, x_i', y_i')$ in a measurement space \mathbb{R}^4 . The corrected measurement denoted $\hat{\mathbf{X}}_i(\hat{x}_i, \hat{y}_i, \hat{x}_i', \hat{y}_i')$ lies on a variety v_H defined by the constraint in (3). $d(\cdot, \cdot)$ is the Euclidean distance between the points. The task of minimizing the 2D reprojection error is to find a variety v_H and $\{\hat{\mathbf{X}}_i\}$ which are closest points on the variety v_H to $\{\mathbf{X}_i\}$.

The 2D reprojection error is proven to be more superior than other geometric error. The minimization of 2D reprojection error is often applied in evaluation step 1(c). But it is usually not applied in other steps of the algorithm. In our algorithm, we will solve the fundamental matrix by minimizing the 2D reprojection error.

The problem of the fundamental matrix estimation can be described as

$$\min_{\mathbf{F}} \sum_{i=\{a \text{ subset of data}\}} d(\mathbf{x}_i, \hat{\mathbf{x}}_i)^2 + d(\mathbf{x}_i', \hat{\mathbf{x}}_i')^2 \quad \text{subject to} \quad \hat{\mathbf{x}}_i' \mathbf{F} \hat{\mathbf{x}}_i = 0 \quad \forall i. \quad (4)$$

We use the factorization method of Tang and Hung [11] to solve this problem. The method iteratively minimizes the 2D reprojection error by evaluating estimated camera matrix, 3D points and projective depth each at a time.

Given an estimate of the fundamental matrix, we will evaluate the consistency of correspondence $(\mathbf{x}_i, \mathbf{x}_i')$ in S with \mathbf{F} by the optimal triangulation method

$$\min_{\mathbf{x}_i, \mathbf{x}_i'} d(\mathbf{x}_i, \hat{\mathbf{x}}_i)^2 + d(\mathbf{x}_i', \hat{\mathbf{x}}_i')^2 \quad \text{subject to} \quad \hat{\mathbf{x}}_i' \mathbf{F} \hat{\mathbf{x}}_i = 0. \quad (5)$$

Optimal triangulation [10] is a linear triangulation method which converts the least square functions to one parameter functions and finds the global minimum.

3.2 Mixture model parameters estimation

A scoring function is introduced to evaluate how the fundamental matrix fits with the data set (more exactly inliers) [8]. The scoring function is defined as:

$$C = \sum_i \rho(e_i^2) \quad \text{where} \quad \rho(e_i^2) = \begin{cases} e_i \cdot \sigma & \frac{e_i^2}{\sigma^2} \leq T \\ \sqrt{T} \cdot \sigma^2 & \frac{e_i^2}{\sigma^2} > T \end{cases} \quad (6)$$

with a modification on the scorings compared with the one in [8].

In (6), T is a threshold for discriminating outliers from inliers, defined by

$$T = 2 \log\left(\frac{1-\gamma}{\gamma}\right) + (D-d) \log\left(\frac{L^2}{2\pi\sigma^2}\right) \quad (7)$$

where L^2 represents the size of the search window for performing matches in the two images, D is the dimension of point correspondences $\{\mathbf{X}_i\}$ and d is the dimension of the variety v_H . Note that the contribution of each inlier to the scoring function depends on its error whereas each outlier increases the scoring function by a constant.

The scoring function performs well when all the components $\{e_i^2\}, \sigma$ and γ are estimated correctly. In the previous section, we have considered how the 2D reprojection error $\{e_i^2\}$ may be minimized. In this section, we will propose a new iterative estimator of σ and γ .

First, we will model the probability distribution of 2D reprojection errors for both inliers and outliers of the data set as a mixture density [12]. We assume that the noise of inliers is a zero-mean Gaussian distribution, whereas the matching error for outliers is a uniform distribution. Thus, the error for each correspondence in the entire data set is a multivariate mixture model of the Gaussian and uniform distribution in D dimensions given by

$$p(e_i | \sigma) = \gamma \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^D \exp\left(-\frac{e_i^2}{2\sigma^2}\right) + (1-\gamma) \frac{1}{v} \quad (8)$$

In (8), γ is the prior probability of the inliers occurring, σ is the standard deviation of the Gaussian distribution on each coordinate of point correspondences, and $\{e_i^2\}$ are the 2D reprojection errors associated with point correspondences in the data

set, which can be regard as unlabeled samples drawn independently from the mixture density. v indicates the volume of the bounded space within which outliers are expected to fall uniformly. Put $D=4$ as $\{e_i^2\}$ is observed in \square^4 . Regarding the state-conditional probability function (8) as a likelihood function and then by maximizing the likelihood of the mixture model, σ and γ can be estimated using an iterative method. Suppose $(\sigma_k, \hat{\gamma}_k)$ are already available, $(\sigma_{k+1}, \hat{\gamma}_{k+1})$ can be computed using the following recursive formulas:

$$\hat{\gamma}_{k+1} = \frac{1}{n} \sum_i z_i \tag{9}$$

$$\text{where } z_i = \frac{\hat{\gamma}_k \left(\frac{1}{\sqrt{2\pi}\sigma_k} \right)^D \exp\left(-\frac{e_i^2}{2\sigma_k^2} \right)}{\hat{\gamma}_k \left(\frac{1}{\sqrt{2\pi}\sigma_k} \right)^D \exp\left(-\frac{e_i^2}{2\sigma_k^2} \right) + (1-\hat{\gamma}_k) \frac{1}{v}} \tag{10}$$

$$\text{and } \sigma_{k+1} = \sqrt{\frac{\sum_i (z_i \cdot e_i^2)}{D \sum_i z_i}} = \sqrt{\frac{\sum_i (z_i \cdot e_i^2)}{n \hat{\gamma}_{k+1} D}} \tag{11}$$

Algorithm 1

The algorithm of mixture model parameters estimation can be summarized in following steps:

1. Given a fundamental matrix F , the optimal triangulation is performed to obtain the 2D reprojection errors $\{e_i^2\}$ for the entire data set.
2. Set the initial estimate $\gamma_0 = \frac{1}{2}$ and $\sigma_0 = \sqrt{\frac{\text{median}(e_i^2)}{D}}$
3. Estimate posterior probability $\{z_i\}$, $i=1, \dots, n$ for the data set from the current estimate γ_k and σ_k using (10).
4. Make a new estimation of γ_{k+1} (9) and σ_{k+1} (11) from the current Maximum likelihood estimation $\{z_i\}$.
5. Step 3 and step 4 are repeated until both γ_{k+1} and σ_{k+1} are convergent (typically need five times).

3.3 Algorithm summary

Based on the above discussion on the determination and evaluation of the fundamental matrix and the mixture model parameter estimation, we suggest a new robust algorithm for computing fundamental matrix. The objective of the algorithm is to determine the maximum likelihood estimation of the fundamental matrix which is optimal under the assumption that inliers obey Gaussian distribution and outliers obey uniform distribution.

Algorithm 2

The algorithm of robust estimation of the fundamental matrix

1. Randomly sample a minimum subset of point correspondences $S_j = \{(\mathbf{x}_i, \mathbf{x}_i') | i = 1, \dots, n\}$ and estimate \mathbf{F}_j by factorization method of [11].
2. Perform the projective reconstruction to obtain $\{\mathbf{X}_i\}$ in 3D space and calculate the 2D reprojection error $\{e_i^2\}$ for the data set by the optimal triangulation.
3. Use Algorithm 1 to estimate the percentage of inliers γ and the standard deviation σ of measurement noise on each coordinate of image points.
4. Compute the threshold T by (7) to discriminate inliers and outliers.
5. Compute the scoring function for \mathbf{F}_j using the equation (6).
6. Go to step1 until N trials. Update N by (2) using γ if necessary.
7. Select the best solution for which the scoring function has the smallest value.

4 Experimental Results

Two sets of experimental results will be given to illustrate how Algorithm 1 (for mixture model parameter estimation) and Algorithm 2 (for robust estimation of the fundamental matrix incorporation the mixture model parameter estimation) work.

4.1 Experiment on parameter estimation algorithm

In this experiment, Algorithm 1 for estimating the parameters of a mixture model is tested. The experimental results are derived from independent tests on 100 sets of 200 point correspondences with varying percentages of outliers. Gaussian noise is added to each coordinate of the image points. The standard deviation of noise is from 0.5 to 3 pixels (in 600×800 images) with 0.5 pixel increment. The ground truth fundamental matrix is used in Step 1 of Algorithm 1 to compute 2D reprojection error, from which the percentage of outliers and noise level are estimated using Algorithm 1. The mean and standard deviation of the percentage errors in the estimation of γ are given in Fig. 1.

For the purpose of comparison, the result of estimating γ using the MLESAC method of [7] is given in Fig. 2. We note that the method used in MLESAC requires

an exact estimate of noise level to achieve good results. Our method performs noticeably better than the MLESAC method, reducing the percentage error in the estimated parameter γ by a factor of 1.5 in most cases.

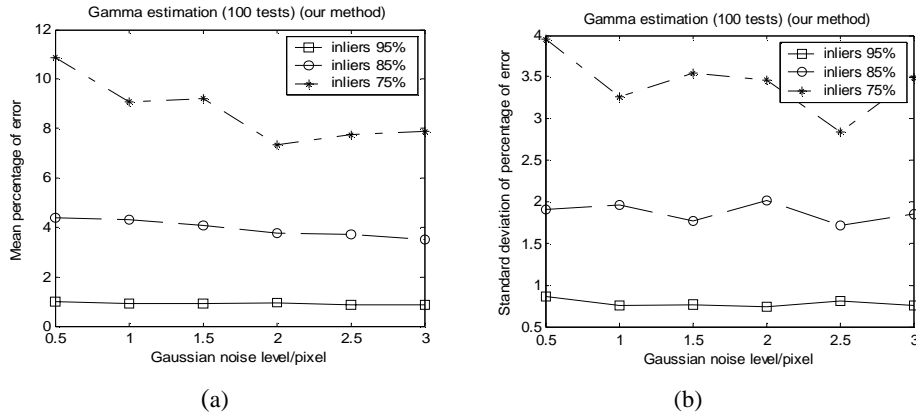


Fig. 1. (a) The mean and (b) the standard deviation of percentage error in the estimated γ (relative to ground truth) for our method.

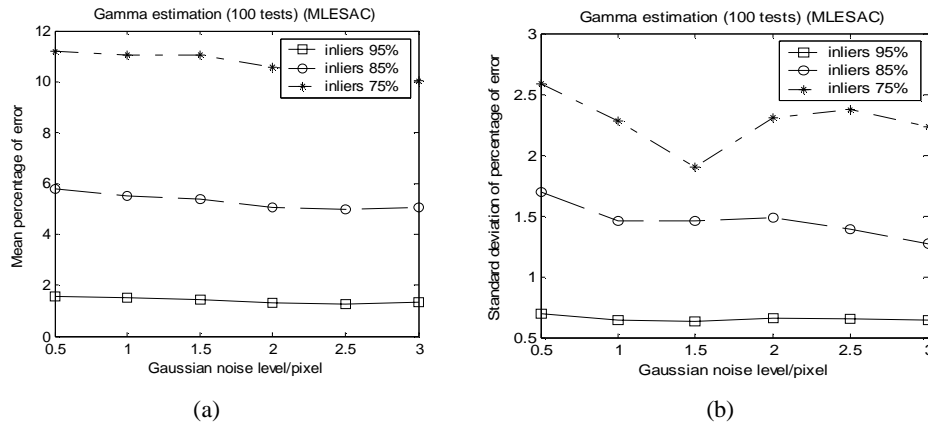


Fig. 2. (a) The mean and (b) the standard deviation of percentage error in the estimated γ (relative to ground truth) for MLESAC method.

4.2 Experiment on our robust method

In this experiment, the performance of Algorithm 2 is tested using synthetic data. The accuracy of the solution $\hat{\mathbf{F}}$ will be assessed using the Sampson distance measure [2]:

$$d_{sampler} = \frac{1}{n} \sum_i \frac{(\mathbf{x}_i^T \hat{\mathbf{F}} \mathbf{x}_i)^2}{(\hat{\mathbf{F}} \mathbf{x}_i)_1^2 + (\hat{\mathbf{F}} \mathbf{x}_i)_2^2 + (\hat{\mathbf{F}}^T \mathbf{x}_i)_1^2 + (\hat{\mathbf{F}}^T \mathbf{x}_i)_2^2} \quad (12)$$

where $(\hat{F}\underline{x}_i)_j$ refers to the j -th entry of $\hat{F}\underline{x}_i$. The underlined symbol \underline{x}_i indicates the noise-free data.

The experimental results are derived from independent tests on 50 sets of 500 point correspondences. The image size is 600×800 . On each coordinate, the standard deviation of Gaussian noise σ is 1 pixel. The percentage of outliers $1-\gamma$ is increased from 5% to 30% in increments of 5% steps. Our method is compared with MAPSAC. The simulation results show that our method performs noticeably better than MAPSAC method though both methods have a small proportion (about 6%) of failures in reconstructing acceptable solutions. These bad solutions have been excluded from the calculation of the mean Sampson distance shown in Figure 3.

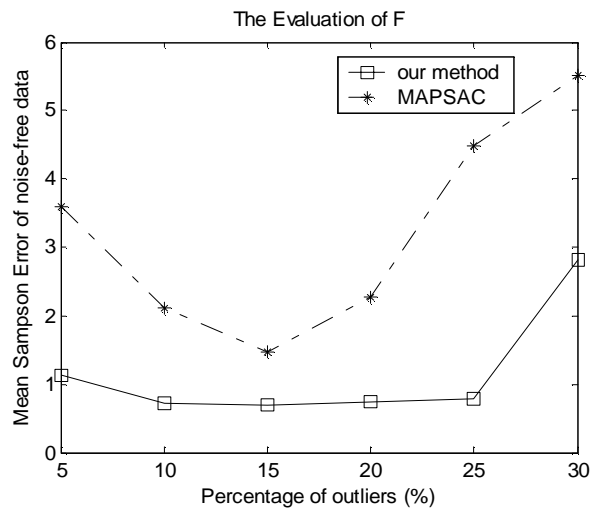


Fig. 3. The mean of the Sampson distance of the solutions estimated by our method and MAPSAC (after eliminating 3 bad solutions among total 50 solutions).

5 Conclusion

In this paper, a robust method has been developed to give an accurate estimation of the fundamental matrix in the presence of outliers. Our method differs from existing method in two ways. First, we minimize the 2D reprojection error in both the determination and evaluation of the fundamental matrix, in contrast to other methods which do not necessarily use a consistent measure throughout the entire process. Secondly, we estimate both unknown parameters γ and σ in a mixture model for image errors of inliers and outliers. These mixture model parameters are estimated together within an iterative algorithm, which provides better results than existing ones.

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