TONES AND CHIRPS DETECTION USING SUMS OF CONJUGATE PRODUCTS

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ABSTRACT

We analyse a class of simple statistics, which we call sums of conjugate products, for the purpose of detecting tones and chirps in noise. These statistics have been studied before for their powers of estimation. In this paper, we derive their asymptotic statistics and compare their discriminating power against the energy detector. We find that a modest improvement over the energy detector is achieved for tone detection, but that the energy detector has superior performance in the case of chirps. Numerical results are presented, which are found to be in good agreement with the theoretical predictions.

1. INTRODUCTION

In this paper, we examine the application of a number of simple detectors which belong to a class which we call sums of conjugate products to the detection of tones and chirps in additive white Gaussian noise. The detection of these signals is fundamental to signal processing, as they occur in diverse fields including telecommunications, radar and sonar.

For a set of complex data, \( \{z_n\}, n = 1, 2, \ldots, N \), containing a noisy record of a tone or chirp, we propose the modulus of the statistics

\[
s = \sum_{n=1}^{N-1} w_n z_{n+1} z_n^* \quad \text{or} \quad s = \sum_{n=1}^{N-2} w_n z_{n+2} z_{n+1}^* z_n^* \quad (1)
\]

for detection of the tone or chirp, respectively, where the \( w_n \) are coefficients (or weights) which are chosen appropriately. We call these statistics sums of conjugate products. They are appealing because of their simplicity and because of the speed with which they can be calculated. For detection of radar returns, the statistic for tones in (1) with \( w_n = 1 \) has been studied since 1960 [1]. A detailed examination was undertaken in [2].

The argument of these statistics, \( \angle s \), can be used to estimate frequency or chirp rate. In this capacity, they have been well-studied in the literature. For frequency estimation, the first form of \( s \) in (1) was studied in [2] with \( w_n = 1 \), where it was known as the simple semicoherent statistic. For different choices of \( w_n \), the statistic has also been studied in [3] as the weighted linear predictor and in [4].

2. SIGNAL MODELS

We will now describe the statistical models we will use to describe the noisy recordings of tone and chirp signals. We assume in each case that we record complex samples \( z_n \), \( n = 1, 2, \ldots, N \), which are instances of complex random variables \( Z_n \) where

\[
Z_n = A \exp[ i(\theta + 2 \pi fn)] + \Gamma_n \quad (2)
\]

in the case of a tone signal or

\[
Z_n = A \exp[ i(\theta + 2 \pi fn + \pi \alpha n^2)] + \Gamma_n. \quad (3)
\]

In each case, \( A \) and \( \theta \) are parameters representing the amplitude and initial phase of the signals. The parameter \( f \) represents the frequency, or initial frequency in the case of the chirp, where \( \alpha \) represents the chirp rate. The \( \Gamma_n \) are independent, identically-distributed (i.i.d.) complex normal random variables with variance \( \sigma^2 \), by which we mean that \( \text{E}[|\Gamma_n|^2] = 2\sigma^2 \).

Although the forms of (2) and (3) make clear the additive nature of the noise, as we would expect in many real systems as a result of thermal noise, we will find it convenient to express the random variables \( Z_n \) as

\[
Z_n = \exp[i(\theta + 2 \pi fn)](A + \Xi_n) \quad (4)
\]
for the tone and chirp cases, respectively. The $\Xi_n$ and the
$\Gamma_n$ are related through multiplication by an exponent with
purely imaginary argument. Therefore, the $\Xi_n$ are also i.i.d.
complex normal random variables with variance $\sigma^2$.

The statistical test which we wish to perform on the
data is to test the null hypothesis that $A = 0$ (noise only)
against the alternative hypothesis that $A > 0$ (signal plus
noise). The statistics we shall use for this purpose are de-
scribed in the next section.

3. DETECTION STATISTICS

In this paper, we examine the statistics

$$s_{UT} = \sum_{n=1}^{N-1} \bar{z}_{n+1}^* z_n^*$$

and

$$s_{WT} = \sum_{n=1}^{N-1} n(N-n)z_{n+1}^* z_n^*$$

and

$$s_{UC} = \sum_{n=1}^{N-2} (n+2)(n+1)(N-n)(N-n-1)$$

Evaluating these statistics makes use of two subscript
letters: the first denotes whether it is unweighted (U) or
weighted (W), the second denotes whether it is intended
for tone signals (T) or for chirps (C).

The statistic $s_{UT}$ is the simple semicoherent statistic of
[2] and (after taking the complex argument) the unweighted
linear predictor frequency estimator of [3]. Similarly, $s_{WT}$
is the weighted linear predictor frequency estimator. The
arguments of $s_{UC}$ and $s_{WC}$ are the unweighted and weighted
linear predictor chirp estimators of [7].

The arguments of the weighted statistics are known [3,
7] to provide more accurate estimates of frequency or chirp
rate than the unweighted statistics for high signal-to-noise
ratios. One might expect that the modulus of these statistics
would provide detection statistics. We will analyse the
performance of these statistics for detection in the next
section.

For the sake of comparison, we will compare each of the
detectors against the energy detector,

$$s_{E} = \sum_{n=1}^{N} |z_n|^2.$$
Let us now calculate the moments for the chirp detectors of (8) and (9). Redefine the random variable

\[ S = \sum_{n=1}^{N-2} w_n Z_{n+2} Z_{n+1}^* Z_n^* \]

where the \( w_n \) are again weights and the \( Z_n \) are random variables defined in (5). By substitution, we find that

\[ S = e^{i2\pi \omega} \sum_{n=1}^{N-2} w_n (A + \Xi_{n+2})(A + \Xi_{n+1})^2(A + \Xi_n) \]

\[ = e^{i2\pi \omega} R, \]

where \( R \) is now redefined. We find that

\[ \mu_R = A^4 \sum_{n=1}^{N-2} w_n, \]

\[ \eta_{RX}^2 = (6A^6 \sigma^2 + 22A^4 \sigma^4 + 32A^2 \sigma^6 + 16\sigma^8) \sum_{n=1}^{N-2} w_n^2 \]

\[ + (4A^6 \sigma^2 + 16A^4 \sigma^4) \sum_{n=1}^{N-3} w_n w_{n+1} \]

\[ + 2A^6 \sigma^2 \sum_{n=1}^{N-4} w_n w_{n+2} \]

and

\[ \eta_{RY}^2 = \eta_{RX}^2 - (8A^6 \sigma^2 + 32A^4 \sigma^4) \sum_{n=1}^{N-3} w_n w_{n+1} \]

For the case of the unweighted chirp detector of (8), this reduces to

\[ \mu_R = A^4 (N-2), \]

\[ \eta_{RX}^2 = (6A^6 \sigma^2 + 22A^4 \sigma^4 + 32A^2 \sigma^6 + 16\sigma^8)(N-2) \]

\[ + (4A^6 \sigma^2 + 16A^4 \sigma^4)(N-3) + 2A^6 \sigma^2(N-4) \]

and

\[ \eta_{RY}^2 = \eta_{RX}^2 - (8A^6 \sigma^2 + 32A^4 \sigma^4)(N-3) \]

We omit expressions for the covariances of the weighted chirp detector of (9) because they are somewhat unwieldy.

Notice that \( \eta_{RX}^2 = \eta_{RY}^2 \) when \( A = 0 \) in each case for both the tone and chirp detectors. We label its value \( \eta_{R0}^2 \) in this case. For tones we have

\[ \eta_{R0}^2 = 2\sigma^4 \sum_{n=1}^{N-1} w_n^2 \]

and for chirps we have

\[ \eta_{R0}^2 = 32\sigma^8 \sum_{n=1}^{N-2} w_n^2 \]

We now adapt the arguments of Lank et al. [2] to compare the asymptotic performance of these detectors with the energy detector. The comparison will be made using the Neymann-Pearson criterion as its basis. The arguments we make are equally applicable to the tone and chirp detectors and so they will be expressed in terms of the random variable \( R \), as it is variably defined.

The central limit theorem of Hoeffding & Robbins [8] can be adapted to show that the distribution of \( R \) is asymptotically normal with means and covariances as described above. As \( N \to \infty \), the distribution of \( |R| \) under the null hypothesis approaches a Rayleigh distribution.

By considering the expression for the probability of false alarm, \( P_{FA} \), and detection, \( P_D \), and holding these two quantities fixed along with \( \sigma^2 \), we can show that \( A \to 0 \) as \( N \to \infty \). Therefore, the covariances \( \eta_{RX}^2 \) and \( \eta_{RY}^2 \) approach \( \eta_{R0}^2 \) and it can then be shown that the distribution of \( |R| \) under the alternative hypothesis approaches a Rician distribution.

Approximate expressions for \( P_{FA} \) and \( P_D \) for a threshold \( d \) under these conditions are then

\[ P_{FA} \approx \exp \left( \frac{-d^2}{2\eta_{R0}^2} \right) \]

and

\[ P_D \approx Q \left( \frac{\mu_R}{\eta_{R0}} \frac{d}{\eta_{R0}} \right) \]

where \( Q(\alpha, \beta) \) is Marcum’s Q-function which is defined as

\[ Q(\alpha, \beta) = \int_{\beta}^{\infty} x \exp \left( -\frac{1}{2} (x^2 - \alpha^2) \right) I_0(x\alpha) \, dx. \]

Under the further condition that \( P_{FA} \) is small, we can usefully equate the values of \( P_{FA} \) and \( P_D \) of the chirp or tone statistic under test with that of the energy detector to discover a relationship between the SNRs and numbers of points required. Using the subscripts \( E \) and \( R \) everywhere to denote quantities associated with the energy detector and the tone or chirp detector under test, respectively, and defining the SNR, \( \rho \), as \( A^2/2\sigma^2 \), it can be shown that

\[ \rho_E \sqrt{N_E} \approx \frac{\mu_R}{\eta_{R0}}. \]

For the unweighted tone detector, this reduces to

\[ \rho_E \sqrt{N_E} \approx \rho_R \sqrt{2N_R} \]

(making, as stated above, the assumption that \( N_R \) is large). This is the asymptotic result of Lank et al. [2]. Thus, with \( N_E = N_R \), the SNR required to obtain the same discriminating power by the energy detector is \( \sqrt{2} \) times higher (or 1.5 dB higher) than that required by the unweighted tone detector.

For the weighted tone detector, we find

\[ \rho_E \sqrt{N_E} \approx \rho_R \sqrt{\frac{5N_E}{3}}. \]

Thus, with \( N_E = N_R \), the SNR required to obtain the same discriminating power by the energy detector is \( \sqrt{5/3} \) times higher (or 1.1 dB higher) than that required by the weighted tone detector.
On the other hand, for the chirp detectors we have

\[
\rho_E \sqrt{N_E} \approx \frac{\rho_R^2 \sum_{n=1}^{N_R-2} w_n}{\sqrt{2} \sum_{n=1}^{N_R-2} w_n^2}
\]

For the unweighted chirp detector, we therefore have

\[
\rho_E \sqrt{N_E} \approx \frac{\rho_R^2 \sqrt{N_R}}{2}.
\]  (12)

This implies that, with \(N_E = N_R\), the SNR required to obtain the same discriminating power by the energy detector is \(\rho_R/\sqrt{2}\) times higher than that required by the unweighted chirp detector. However, as we found above, \(\rho_R \to 0\) as \(N_R \to \infty\). Therefore, the unweighted chirp detector has increasingly poorer discriminating power, relative to the energy detector, as the number of points is increased.

The situation for the weighted chirp detector is even worse (although only slightly). For the weights

\[
w_n = n(n+1)(N-n-1)(N-n),
\]

we find that

\[
\lim_{N \to \infty} \frac{\sum_{n=1}^{N-2} w_n}{\sqrt{N \sum_{n=1}^{N-2} w_n}} = \sqrt{\frac{7}{10}}.
\]

Hence, we have

\[
\rho_E \sqrt{N_E} \approx \rho_R \sqrt{\frac{7N_R}{20}}.
\]  (13)

Therefore, the SNR of the weighted chirp detector is worse than the unweighted detector by a factor of \(\sqrt{7/10}\) (or 0.7 dB) for a given discriminating power.

5. NUMERICAL RESULTS

In Figure 1, the results of numerical simulations are presented. They compare the discriminating abilities of the energy detector with the weighted and unweighted tone and chirp detectors. They were computed using a Monte Carlo technique whereby one million trials were conducted to generate the distributions for each SNR value and for each detector. The experiments were conducted with \(N = 24\) and \(P_{FA} = 10^{-4}\).

Although the choice of \(N = 24\) would seem to be much less than necessary to approach the asymptotic behaviour analysed in the previous section, we still see quite good agreement with the theoretical predictions. Measuring the offset of the curves, we see that the curve for the unweighted tone detector is offset by approximately 1.1 dB to the left of the energy detector, whereas (10) predicts 1.5 dB. The offset of the weighted tone detector is approximately 0.7 dB whereas (11) predicts 1.1 dB. The offset of the unweighted chirp detector is approximately 1.7 dB to the right of the energy detector, whereas (12) predicts 2.0 dB. The weighted chirp detector is offset a further 0.7 dB to the right of the unweighted chirp detector which agrees with the theoretical prediction. Overall, we find there is quite good agreement between the theoretical predictions and the observations from experiment presented in Figure 1.

6. REFERENCES


