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GROUPS OF EXPONENT EIGHT

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CONTENTS

This is a report (finalized at this meeting) of the first steps in a study of groups of exponent 8 aimed at answering the special case of Burnside's question: is every 2-generator group of exponent 8 finite? It contains the bare details of the work. More informal accounts were given in lectures to the workshop of which written versions appear elsewhere in these proceedings.

INTRODUCTION

One of the driving forces behind the study of the structure of groups is a number of questions posed by Burnside (1902). In particular he asked whether every m-generator group of exponent n (every element has order dividing n) is finite. It is known that the answer is 'yes' for exponents 2, 3, 4, 6 and all (finite) numbers of generators (see, for example, Sections 5.12 and 5.13 of Magnus, Karrass, Solitar (1966)) and 'no' for all *odd* exponents greater than or equal to 665 (see Adian (1974), (1975)).

These results focus particular attention on the exponents which are powers of two. This paper is a report on some first steps in a study of groups of exponent eight.

We denote the free group of rank m and exponent n by $B(m, n)$.

Little seems to be known about groups of exponent 8. In 1947 Sanov showed that there is a 2-generator group of exponent 8 with order 2^{136} by making use of the Schreier formula for the rank of a subgroup of finite index in a free group of finite rank. It is part of the folk-lore that the same argument applied to the subgroup F^4 generated by fourth powers of elements in a free group F of rank 2 shows that $F/\left(F^4\right)^2$ is a 2-generator group of exponent 8 and order 2^{4109} (because F/F^4 has order 2^{12} - see §6.8 of Coxeter and Moser (1957)). Recently Shield (1977) has shown that the nilpotency class of $F/\left(F^4\right)^2$ is 39. In 1951 Sanov began another area of investigation by proving that all groups of exponent 8 satisfy the 23rd Engel congruence. The best result in this direction is due to Krause (1964); he showed they satisfy the 14th Engel congruence. Macdonald ((1973), p. 437), using a computer implementation of an algorithm for calculating nilpotent quotient (or factor) groups, showed that the largest 2-generator group of exponent 8 and nilpotency class 6 has order 2^{35}. This has since been extended to class 12 and order 2^{722} using the Canberra implementation of a related algorithm called the *nilpotent quotient algorithm* (see Newman (1976)). Hermanns (1977) has recently studied the largest metabelian quotient group of $B(2, 8)$ and has shown that it has order 2^{63} and nilpotency class 12 (this has been confirmed by the nilpotent quotient algorithm).

The present study was begun by two of us (Grunewald, Mennicke). It has its origin in the possibility that the nature of the subgroups of $B(2, 8)$ might give a hint as to whether to expect that $B(2, 8)$ is finite or not. Adian in ((1971), Corollary 2) proved that $B(m, n)$ has all its finite subgroups cyclic for n odd and at least 4381 (the bound has since been reduced by him to 665 ; (1975), Theorem VII.1.8). This might suggest that a pointer to finiteness of $B(2, 8)$ would be obtained if it were possible to show that $B(2, 8)$ has comparatively large finite subgroups. However the situation is not quite so simple because every infinite 2-group has an infinite abelian subgroup (see Theorem 3.41 of Robinson (1972)). Therefore infinite 2-groups have abelian subgroups of every 2-power order. In the context of groups of exponent 8 these large finite abelian subgroups require many generators. This suggests that a pointer to finiteness of $B(2, 8)$ would be obtained if one could show that $B(2, 8)$ has comparatively large finite 2-generator subgroups. Whatever one feels about such motivation, it raises finiteness problems which are more approachable than that for $B(2, 8)$ itself. Some of these are discussed below. For those with affirmative solution examples (described later) give a lower bound for the order of the subgroup of $B(2, 8)$ in question. As Burnside himself pointed out, once one knows that a group is finite the next problem is to determine its order. This is discussed for some of the groups which arise.

Let $\{a, b\}$ be a free generating set for $B(2, 8)$ - from now on simply B . Before turning attention to 2-generator subgroups of $B(2, 8)$ we look briefly at 1-generator or cyclic subgroups, in part to prepare for some questions which arise in the 2-generator context. Here, of course, there is no finiteness problem and the order of every cyclic subgroup or element divides 8 . An element which has odd exponent sum in either a or b has order 8 for it has order 8 in a cyclic homomorphic image. Equally clearly squares of elements have order at most 4 . For some other elements it is comparatively easy to determine the order by constructing suitable examples. One sees that a^2b^2 and the commutator $[b, a]$ have order 8 by considering the group presented by

$$\langle k, h \mid k^8 = h^4 = 1, k^{h^2} = k^{-1}, kk^h = k^hk \rangle .$$

This group is a splitting extension of a direct product of two cyclic groups of order 8 (the normal closure of k) by a cyclic group of order 4 (generated by h). It is easily seen to have exponent 8 , so the mapping $a \mapsto hk$, $b \mapsto h$ can be extended to a homomorphism from B . Under this homomorphism $\left(a^2b^2\right)^4$ and $[b, a]^4$

both map to $(kk^h)^4$ which is not the identity. Considering the group K presented by

$$\langle k, h \mid k^8 = h^8 = 1, k^{h^4} = k^{-1}, kk^h = k^h k, kk^{h^2} = k^{h^2} k, kk^{h^3} = k^{h^3} k\rangle ,$$

which is a splitting extension of a direct product of four cyclic groups of order 8 by a cyclic group of order 8 and has exponent 8, settles the order problem for more complicated elements; for instance, $a^4 b^4$ and $[b, a, a, a]$ both have order 8. For commutators of higher weight we record a few coarser results. Let B_c denote the cth term of the lower central series of B and B'' the second derived group of B. The work of Hermanns (1977) shows that $(B_5)^4 \le B''$ and $(B_9)^2 \le B''$. Calculating in B/B_{12} gives $(B_6)^2 \le B_7$ (*cf.* p. 437 of Macdonald (1973)), $(B_5)^4 \le B_{12}$ and $(B_7)^2 \le B_{11}$. Hence, for example, $(B_c)^2 \le B_{c+1}$ for all c greater than or equal to 6. Finally we mention, for later use, that $(a^4 b^4)^4$ lies in B_{12} so that $[(a^4 b^4)^4, b^2, (a^4 b^4)^4]$ is in B_{26}.

We now turn to 2-generator subgroups of B. Observe first that if the two generators have order 2, then the subgroup they generate is dihedral of order 4, 8 or 16. The first case of interest is $\langle a^4, b^4\rangle$ (here $\langle\ \rangle$ denotes subgroup generated by). This has order 16; consider $\langle (hk)^4, h^4\rangle$ in the group K above. If either generator does not have order 2 finiteness problems immediately arise. Even the simplest seems difficult.

PROBLEM 1. *Is the subgroup of* $B(2, 8)$ *generated by* $\{a^4, b^2\}$ *finite?*

Since the paper was submitted it has been shown (by Havas and Newman) that this subgroup has a largest finite quotient - of order at most 2^{205} and class at most 26. The subgroup $\langle a^4, b^2\rangle$ may be regarded as a product of its finite subgroups $\langle a^4, b^4\rangle$ and $\langle b^2\rangle$; these meet in $\langle b^4\rangle$ (in other words $\langle a^4, b^2\rangle$ is generated by an amalgam of finite groups). In $\langle a^4, b^4\rangle$ there are two proper subgroups $\langle (a^4 b^4)^4, b^4\rangle$ and $\langle (a^4 b^4)^2, b^4\rangle$ containing b^4. This suggests two intermediate finiteness problems. Is $L = \langle (a^4 b^4)^4, b^2\rangle$ finite? Is $M = \langle (a^4 b^4)^2, b^2\rangle$ finite? The first of these is easy to answer for b^4 is central in L and $L/\langle b^4\rangle$ is generated by two elements of order 2, so L is finite of order at most 32. In more detail L is a homomorphic image of

$$\langle T, Y \mid T^2 = Y^4 = [T, Y^2] = (TY)^8 = 1\rangle$$

under the mapping $T \mapsto (a^4b^4)^4$, $Y \mapsto b^2$. Thus L has order 32 if and only if $[(a^4b^4)^4, b^2, (a^4b^4)^4]$ is non-trivial; so the earlier observation that this element lies in B_{26} implies that, if L has order 32 , an example of nilpotency class at least 26 will be involved in exhibiting this order. The second question is harder. Its solution is the main result of the investigation so far.

The subgroup of $B(2, 8)$ *generated by* $\{(a^4b^4)^2, b^2\}$ *is finite of order at most* 2^{31} .

The best lower bound we have for the order of this subgroup M is 2^5 which comes by considering $\langle ((hk)^4h^4)^2, h^2\rangle$ in the group K .

PROBLEM 2. *What are the orders of the subgroups of* $B(2, 8)$ *generated by* $\{(a^4b^4)^4, b^2\}$ *and* $\{(a^4b^4)^2, b^2\}$ *?*

The proof of the finiteness of M , as can be seen by glancing ahead, is highly computational and combinatorial. In the later stages it has been significantly machine assisted. Below we give an outline of the proof and its development. The proof has evolved in a way which is hard, and not always illuminating, to describe. The account here contains some traces of the evolution. They have been kept partly because their removal would not, we believe, significantly aid understanding (and could introduce errors which hinder it) and partly because their retention may shed some light on what was involved in developing the proof.

Observe to begin with that M is a homomorphic image of the group Γ^* presented by

$$\langle Y, W \mid Y^4 = W^4 = (Y^2W)^2 = 1, \text{ exponent } 8\rangle$$

under the mapping $Y \mapsto b^2$, $W \mapsto (a^4b^4)^2$. We prove that Γ^* is finite of order 2^{31} . Another presentation for Γ^* was actually used, namely

$$\langle Y, Z \mid Y^4 = (YZ)^4 = (YZ^{-1})^2 = 1, \text{ exponent } 8\rangle .$$

A finiteness proof need only use finitely many eighth powers, so it can be set in the context of some suitable finitely presented preimage of Γ^* . The one we ended up with is the group Γ with the presentation given at the beginning of Chapter 2. This group has a subgroup Γ_1 of index 8 generated by

$$\{(ZY)^2, YZ^2Y^{-1} = (YZ)^2Z^{-2}, ZYZ^2Y^{-1}Z^{-1} = (ZY)^2Z^{-2}\}$$

(Theorem 1 of Chapter 2). A major step in the proof is to show $\langle (ZY)^2Z^{-2}, (YZ)^2Z^{-2}\rangle$ is a finite 2-group. This is done by considering a preimage G of the group G^* presented by

$$\langle p,\ q \mid p^4 = q^4 = (pq)^4 = \left(pq^{-1}\right)^4 = \left(p^2q^2\right)^4 = \\ = \left(pqp^{-1}q^{-1}\right)^4 = \left(p^2qpq^2pq\right)^2 = 1,\ \text{exponent } 8\rangle\ .$$

Theorem 3 of Chapter 2 shows that the group G is a preimage of the subgroup via the mapping $p \mapsto (ZY)^2Z^{-2}$, $q \mapsto (YZ)^2Z^{-2}$. (The relation $\left(qp^{-1}qpq^{-1}p\right)^4 = 1$ - p. 126 was found after the proof in Chapter 1 was essentially complete.) The proof of the finiteness of G was the subject of Grunewald's dissertation (1973) and of a lecture (by Mennicke) at Oberwolfach in May 1974. The collaboration reported here arose out of a discussion (between Mennicke and Newman) following that lecture. It seemed that a judicious mixture of hand and machine calculations should result in better progress. This has been the case, firstly because great care and considerable time are needed to try to ensure that hand calculations are accurate, and secondly because the nilpotent quotient algorithm is able to give quite detailed information about the groups involved. A presentation for G is given at the beginning of Chapter 1. In that chapter G is shown to be finite of order at most 2^{21} (Theorem 4). Moreover G has exponent 8 (Theorem 7), so G and G^* coincide. The nilpotent quotient algorithm was applied to the presentation for G . It terminated, showing that G has a largest finite 2-quotient of order 2^{21} and nilpotency class 9 . Combined with the finiteness proof this gives that G^* has order 2^{21} . A consistent power commutator presentation for G^* is given in Chapter 3. (This presentation was used to help check the calculations in Chapter 1.) The nilpotent quotient algorithm also yields that the rank of the multiplicator of G^* is 7 . This suggests the possibility that G^* has a presentation involving only two eighth powers in addition to the explicitly given relations. This has been confirmed by a technique based on coset enumeration. This is described in Chapter 3 and provides an alternative and independent proof for the finiteness of G^* . Chapter 2 contains the rest of the finiteness proof for Γ^* (via one for Γ). This proof involves less detailed calculations than those in Chapter 1 because in the final stages of completing the proof it was known that the nilpotent quotient algorithm applied to Γ^* terminates and shows that Γ^* has a largest finite 2-quotient of order 2^{31} and nilpotency class 19 . The resulting consistent power commutator presentation can be deduced from that for Γ given in Chapter 4. This presentation was used to help guide and check the proof in Chapter 2. From the presentation can be "read off" properties of Γ^* such as $\left((\Gamma^*)^4\right)^2$ is non-trivial and $\left(\left((\Gamma^*)^4\right)^2\right)^2$ is trivial. The group Γ which is shown to be a finite 2-group in Chapter 2, is a proper preimage of Γ^* ; the nilpotent quotient algorithm shows Γ to have order 2^{33} and nilpotency class 21 .

The image of G^* in Γ has order 2^{14} . The kernel is generated as a normal subgroup of G^* by

$$\{[[q, p], (q^2p)^4], [q, p^2]^2[q^2, p]^2, (qpq^{-1}p)^4\} .$$

The elements of the kernel when mapped into B via Γ^* yield words in a, b which may be viewed as non-obvious consequences of the exponent 8 condition. The images of the generators of the kernel are rather unpleasant to write down, some other examples of non-obvious consequences are

$$[(a^4b^4a^4b^2)^2, b^4(a^4b^4a^4b^2)^4b^4]^4 ,$$

$$\left((a^4b^4a^4b^2)^3b^4(a^4b^4a^4b^2)^3b^2(a^4b^4a^4b^2)^2b^2\right)^4 .$$

We are indebted to Mr W.A. Alford of the Australian National University for doing some of the computing and to Mr R.I. Yager, a vacation scholar at the Australian National University during the summer of 1976-1977, for his careful checking of the proofs in Chapter 2 which enabled us to eliminate a number of errors and misprints from a draft.

CHAPTER 1

A FINITENESS PROOF FOR G

We shall prove that the group

$$
\begin{aligned}
G = \langle p, q \mid \quad & \text{(i)} & & p^4 = 1 \\
& \text{(ii)} & & q^4 = 1 \\
& \text{(iii)} & & (pq)^4 = 1 \\
& \text{(iv)} & & (pq^{-1})^4 = 1 \\
& \text{(v)} & & (p^2q^2)^4 = 1 \\
& \text{(vi)} & & (pqp^{-1}q^{-1})^4 = 1 \\
& \text{(vii)} & & (p^2qpq^2pq)^2 = 1 \\
& \text{(viii)} & & (q^2pq^2p^{-1})^8 = (p^2qp^2q^{-1})^8 = 1 \\
& \text{(ix)} & & (p^2q)^8 = (q^2p)^8 = 1 \\
& \text{(x)} & & (pqpq^{-1})^8 = (qpqp^{-1})^8 = 1 \\
& \text{(xi)} & & (p^2qpq^{-1}p^2qp^{-1}q^{-1})^8 = (q^2pqp^{-1}q^2pq^{-1}p^{-1})^8 = 1 \\
& \text{(xii)} & & (qpq^{-1}p^2)^8 = (pqp^{-1}q^2)^8 = 1 \\
& \text{(xiii)} & & (p^2q^2p^2q^{-1})^8 = (q^2p^2q^2p^{-1})^8 = 1 \\
& \text{(xiv)} & & (pqp^2q^2)^8 = (qpq^2p^2)^8 = 1 \\
& \text{(xv)} & & (pqp^2q^{-1})^8 = (qpq^2p^{-1})^8 = 1 \rangle
\end{aligned}
$$

is finite. The elementary computations which follow will also show the structure of this group.

A. The Main Theorems

THEOREM 1. *The commutator subgroup* $G' \le G$ *having the index**

$$|G : G'| = 2^4$$

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* If $H_1 \le H_2$ are two groups we write

$$|H_2 : H_1|$$

for the index of H_1 in H_2.

*is generated by:**

$$(1)\quad a = qpq^{-1}p^{-1} \qquad (4)\quad d = q^2pq^{-2}p^{-1}$$

$$(2)\quad b = p\cdot qpq^{-1}p^{-1}\cdot p^{-1} \qquad (5)\quad e = p\cdot q^2pq^2p^{-1}\cdot p^{-1}\ .$$

$$(3)\quad c = p^2\cdot qpq^{-1}p^{-1}\cdot p^{-2}$$

This theorem is proved by the transformation formulae in Lemma 1.

DEFINITION. $H := \langle a, b, c\rangle$.

Let R_H be a defining system of relations for H . Let $R_{G'}$ be a defining system of relations for G' . Let H_g be the subgroup of H consisting of the words with an even number of letters.

$$G'_d := \mathrm{Ker}\left(G' \to \langle a, b, c, d, e \mid R_{G'};\ a = b = c = 1;\ d^2 = e^2 = (ed)^2 = 1\rangle\right)\ .$$

$$D := d^2 = \left(q^2pq^{-2}p^{-1}\right)^2\ .$$

$$F := ebe^{-1}a^{-1}ba^{-1} = \left(pq^2\right)^4\ .$$

H_g is generated by

$$a^2,\ b^2,\ c^2,\ ab,\ ac,\ bc$$

and has in H the index

$$|H : H_g| \le 2\ .$$

G'_d is generated by

$$a,\ dad^{-1},\ eae^{-1},\ deae^{-1}d^{-1},\ b,\ dbd^{-1},\ ebe^{-1},\ debe^{-1}d^{-1},\ c,\ dcd^{-1},\ ece^{-1},\ dece^{-1}d^{-1},$$
$$d^2,\ ede^{-1}d^{-1},\ dede^{-1},\ e^2,\ de^2d^{-1}$$

and has in G' the index

$$|G' : G'_d| \le 2^2\ .$$

Obviously, the systems of generators stated are reductions of the systems of generators obtained by the Reidemeister-Schreier method.

THEOREM 2. (1) *D normalizes $\langle a, b, c, F\rangle$.*

(2) *$D^2 \in \langle a, b, c\rangle$. Moreover D^2 lies in the centre of G .*

(3) *$G'_d = \langle a, b, c, F, D\rangle$.*

---

* We use here a, b , which were already used for generators of $B(2, 8)$ in the introduction. This should cause no confusion.

(4) $|G' : \langle a, b, c, F, D\rangle| \le 2^2$.

(5) $|\langle a, b, c, F, D\rangle : \langle a, b, c, F\rangle| \le 2$.

(6) $|\langle a, b, c, F\rangle : \langle a, b, c\rangle| \le 2^3$.

Proof. (1), (2) follow by Lemma 16.

(3) follows by Lemma 2 (36)-(48).

(6) Lemma 16 and the transformation formulae derived from Lemmas 13, 15 show that F normalizes the normal subgroup N generated by $a^2, b^2, c^2, ac, (ab)^2$ in $\langle a, b, c\rangle$. Since the elements $FaFa^{-1}$ and $FbFb^{-1}$ lie in the centre of $\langle a, b, c, F\rangle$ by Lemma 16 and since both are of the order 2 , the factor group

$$\langle a, b, c, F\rangle/N$$

is of an order smaller than 2^5 . Since it is obvious that N in $\langle a, b, c\rangle$ has index 4 , the proof of (6) is complete.

THEOREM 3. *$\langle (ac)^2, (c^{-1}a)^2, (abcb^{-1})^2\rangle$ is an elementary abelian subgroup of exponent 2 in the centre of H . The following relations hold in the factor group*

(1) $a^4 = b^4 = c^4 = 1$,

(2) $(ac)^2 = (abcb^{-1})^2 = (c^{-1}a)^2 = (abcb)^2 = 1$,

(3) $(ab)^4 = (bc)^4 = 1$,

(4) $bcab^{-1} = ac$,

(5) $bacb^{-1} = ca$,

(6) $(bcb^{-1}a^{-1})^2 = 1$.

Proof. The first relation follows by Theorem 10, Lemma 11 (28), Lemma 13 (19), Lemma 11 (13), Lemma 16.

(2) follows by Lemma 16.

(3) Lemma 11 (26), (22).

(4) Lemma 11 (24).

(5) Lemma 11 (24), Lemma 4 (1).

(6) Lemma 11 (7).

Obviously, in the group $H/\langle (ac)^2, (c^{-1}a)^2, (abcb^{-1})^2\rangle$, $\langle ac, ca\rangle$ is an elementary abelian normal subgroup of order 2^2 . The factor group is a factor group of the group studied by Coxeter (1940):

$$\langle P,\ Q \mid P^4 = Q^4 = (PQ)^4 = \left(P^{-1}Q\right)^4 = \left(P^{-1}Q^{-1}PQ\right)^2 = \left(PQ^{-1}PQ\right)^2 = \left(P^{-1}QPQ\right)^2 = 1\rangle\ .$$

At most, it is of order 2^6 . So the proof of the following theorem is complete.

THEOREM 4. *G is finite and*

$$|G| \le 2^{21}\ .$$

We shall now describe the normal subgroup which is generated in G by the 4th powers.

DEFINITION.

$$v_1 := b^{-1}a^{-1}Dba = \left(p^2q\right)^4\ ,$$

$$v_2 := ab^{-1}a^{-1}Db\ ,$$

$$v_3 := c^{-1}b^{-1}cb^{-1}D^{-1}cb^{-1}Fac^{-1}b^{-1}a^{-1}b^{-1}Fb\ ,$$

$$v_4 := (ac)^2 = \left(qpq^{-1}p\right)^4\ ,$$

$$v_5 := \left(abcb^{-1}\right)^2\ ,$$

$$v_6 := \left(a^{-1}c\right)^2\ ,$$

$$V := \langle v_1,\ v_2,\ v_3,\ v_4,\ v_5,\ v_6\rangle\ .$$

Let α be the automorphism of G which interchanges p and q .

$$u_i := \alpha\left(v_i\right)\ ,\quad i = 1,\ 2,\ 3,\ 4,\ 5,\ 6\ ,$$

$$U := \langle u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ u_6\rangle\ .$$

THEOREM 5. *V is the normal subgroup generated by $\left(p^2q\right)^4$ in G . U is the normal subgroup generated by $\left(q^2p\right)^4$ in G . $V\cdot U$ is elementary abelian of exponent 2 . $V \cap U \ge \langle v_4v_5,\ v_6\rangle$. $|V\cdot U| \le 2^{10}$.*

Proof. The above statements follow by Lemma 13, Lemma 15, Lemma 16.

$$\begin{aligned} B(2,\ 4) := \langle x,\ y \mid w^4 = 1\rangle \simeq \langle x,\ y \mid x^4 = y^4 = (xy)^4 = \left(x^{-1}y\right)^4 = \left(x^2y\right)^4 \\ = \left(xy^2\right)^4 = \left(x^2y^2\right)^4 = \left(x^{-1}y^{-1}xy\right)^4 = \left(x^{-1}yxy\right)^4 = 1\rangle\ . \end{aligned}$$

This presentation is found in Coxeter-Moxer (1957, p. 81). Now we shall show that

$$\left((xy)^2(yx)^2\right)^2 \in Z\left(B(2,\ 4)\right)\ .$$

It is sufficient to show the identity

$$\left((xy)^2(yx)^2\right)^2 = \left(x^2yxy^2xy\right)^2 \ .$$

this is equivalent to $x \rightleftarrows \left((xy)^2(yx)^2\right)^2$; where $\rightleftarrows$ denotes commutes with. To obtain the commutator relation which remains to be shown, it suffices to exchange x and y in the above identity:

$$\begin{aligned}\left((xy)^2(yx)^2\right)^2\cdot\left(x^2yxy^2xy\right)^2 &= xyxy\cdot yxyx\cdot xyxy\cdot yxyx^{-1}\cdot yxyx^{-1}\cdot xyxy\cdot x\cdot xyxy\cdot yxy \\ &= xyxy\cdot x^{-1}y^{-1}\cdot x^{-1}y^2x^{-1}y^2\cdot x^{-1}y^{-1}xy^{-1}\cdot x^{-1}y^2x^{-1}y^2\cdot x^{-1}y^{-1}x^{-1}yxy \\ &= (xy)^4 = 1 \ .\end{aligned}$$

It is well-known that $\left((xy)^2(yx)^2\right)^2 \neq 1$ in $B(2, 4)$.

A proof of this fact is also given by Theorem 8.

THEOREM 6. *$G/V\cdot U$ is of exponent 4 and*

$$|G/V\cdot U| = 2^{11} \ .$$

Proof. Obviously, it is sufficient to show that the relation

$$\left(pqpq^{-1}\right)^4 = 1$$

holds in $G/V\cdot U$. This follows by the definition of v_4 . When determining the order, remember that $B(2, 4)$ is exactly of order 2^{12} . Thus we have shown that $V\cdot U$ is exactly the normal subgroup of G which is generated by the 4th powers.

Next follows:

THEOREM 7. *G is of exponent 8 ; G^4 is elementary abelian of exponent 2 . For G we have the following diagram:*

$$\begin{tikzcd} & G \arrow[d, dash, "2^{11}"] & & \\ & G^4 = V\cdot U \arrow[dl, dash] \arrow[dr, dash] & & \\ U = \langle\langle \left(q^2p\right)^4\rangle\rangle_G \arrow[dr, dash] & & V = \langle\langle \left(p^2q\right)^4\rangle\rangle_G \arrow[dl, dash] & \leq 2^{21} \\ & V\cap U \geq \langle v_4v_5,\ v_6\rangle \arrow[d, dash, "2^2"] & & \\ & E & & \end{tikzcd}$$

where $\langle\langle x\rangle\rangle_G$ denotes the normal subgroup generated by x in G .

The following computations yield also a description of the commutator subgroup of

$$G/V{\cdot}U = G/G^4 .$$

For the generators of $(G/V{\cdot}U)'$ we shall use the same notation as for the generators of G'.

THEOREM 8. *In* $(G/G^4)'$ *we have the following relations:*

(1) $a^2 = b^2 = c^2$;

(2) $(ab)^2 = (bc)^2 = d^2 = e^2$;

(3) $ac = ca$;

(4) $a^2, e^2 \in Z\big((G/G^4)'\big)$;

(5) $dad^{-1} = e^2a$,
$dbd^{-1} = e^2b$,
$dcd^{-1} = e^2c$;

(6) $eae^{-1} = e^2a$,
$ebe^{-1} = e^2b$,
$ece^{-1} = e^2c$.

Proof. These relations are all stated in Lemma 16. Dividing $(G/G^4)'$ by the normal subgroup $\langle a^2, e^2\rangle$ one obviously obtains an elementary abelian group of order 2^5. This implies also that G/G^4 is of order 2^{11} at most.

THEOREM 9. (1) *The solubility length of* G *is at most* 3 .

(2) *The nilpotency class of* G *is at most* 9 .

Proof. (1) Lemma 16 yields that

$$[G', G'] \leq \langle V{\cdot}U, D\rangle$$

where $\langle V{\cdot}U, D\rangle$ is an abelian group.

(2) Let $\zeta_i(G)$ be the ith term of the upper central series of our group G, that is, $\zeta_0(G) = \langle 1\rangle$ and $\zeta_{i+1}(G)/\zeta_i(G) = Z\big(G/\zeta_i(G)\big)$.

Considering the formulas in Lemmas 13, 15, 16 one finds that $\langle U{\cdot}V, D\rangle \subseteq \zeta_5(G)$. The formulas of Theorem 8 and Lemma 1 then imply:

$$\langle de, a^2, ac, U{\cdot}V, D\rangle \subseteq \zeta_6(G) ;$$

$$\langle d, ab, de, a^2, ac, U{\cdot}V, D\rangle \subseteq \zeta_7(G) ;$$

$$[G, G] \subseteq \zeta_8(G) .$$

This proves (2).

B. The Proofs

LEMMA 1. *We have the following rules of conjugation:*

(1) $pap^{-1} = b$, (6) $qaq^{-1} = da^{-1}$,

(2) $pbp^{-1} = c$, (7) $qbq^{-1} = aeb^{-1}a^{-1}$,

(3) $pcp^{-1} = c^{-1}b^{-1}\;{}^{-1}$, (8) $qcq^{-1} = adbc^{-1}b^{-1}a^{-1}$,

(4) $pdp^{-1} = e$, (9) $qdq^{-1} = c^{-1}e^{-1}a^{-1}$,

(5) $pep^{-1} = b^{-1}db$, (10) $qeq^{-1} = ad^{-1}abcb^{-1}a^{-1}$.

Proof. (1) to (4) follow by the definition of $a - e$ using (i), (ii).

(5) follows by (i), (ii), (iii), (iv).

(6) and (7) follow by the definition of $a - e$.

(8) By the definition of $a - e$ and (i), (ii) we have

$$qcq^{-1}abcb^{-1}d^{-1}a^{-1} = qp^2qpq\,\left(qp^{-1}\right)^4(pq)^4q^{-1}p^{-1}q^{-1}p^{-2}q^{-1} \ .$$

Then (iii), (iv) imply (8).

(9) follows immediately by (ii), (iv).

(10) follows by (iii).

LEMMA 2. *In* $\langle a, b, c, D, F\rangle$ *we have the relations:*

(1) $a^4 = b^4 = c^4 = D^4 = F^2 = 1$;

(2) $(ac)^4 = (abc)^4 = \left(abcb^{-1}\right)^4 = \left(ca^{-1}\right)^8 = 1$;

(3) $\left(Dbc^{-1}bc\right)^2 = 1$;

(4) $\left(Dbc^{-1}\right)^2 = (ab)^2$;

(5) $d \rightleftarrows abcb^{-1}$, $D \rightleftarrows abcb^{-1}$;

(6) $c^2 \rightleftarrows b^{-1}a^{-1}Dba$;

(7) $\left(D^{-1}ab\right)^2 = \left(bc^{-1}\right)^2$;

(8) $\left(Dbab^{-1}a^{-1}\right)^2 = 1$;

(9) $\left(Dbaba^{-1}\right)^2 = 1$;

(10) $Dbacb^{-1}D^{-1}aba^{-1}c^{-1}b^{-1}a^{-1} = 1$;

(11) $Dbc^{-1}bDb \rightleftarrows abca^{-1}$;

(12) $abc^{-1}b^{-1}a^{-1}Dbc^{-1}bca^{-1}b^{-1}D^{-1}b^{-1}ca = 1$;

(13) $(D^{-1}abca^{-1}Db^{-1}c^{-1})^2 = 1$;

(14) $(Dba^2b)^4 = 1$;

(15) $(Dba^2bc^{-1}bc^{-1}b)^2 = 1$;

(16) $(c^{-1}b^{-1}a^{-1}Dca^{-1})^4 = 1$;

(17) $b^2 \rightleftarrows (c^{-1}b^{-1}a^{-1}Dca^{-1})^2$;

(18) $(DbaDbc^{-1}bDbac^{-1})^2 = 1$;

(19) $Fa^2F = a^2$;

(20) $Fbc^2b^{-1}F = bc^2b^{-1}$;

(21) $Fab^2c^2aF = a^{-1}c^2b^2a^{-1}$;

(22) $F(ab^{-1})^2F = a^{-1}c^2b^{-1}a^{-1}c^2b^{-1}$;

(23) $Fab^{-1}abca^{-1}F = a^{-1}c^{-1}abcacb^{-1}$;

(24) $cb^{-1}Fac^{-1}b^{-1}a^{-1} \rightleftarrows b^2$;

(25) $c^{-1}bcbcFac^{-1}b^{-1}FDbac^2b^{-1}a^{-1}c^{-1}Fac^{-1}b^{-1}FDbac^2b^{-1}a^{-1} = 1$;

(26) $caFac^{-1}b^{-1}FDbac^2b^{-1}Fac^{-1}b^{-1}FDba^2bc = 1$;

(27) $aca^{-1}b^{-1}D^{-1}Fbca^{-1}Fa^{-1}c^{-1}Fac^{-1}b^{-1}FDba = 1$;

(28) $bc^2a^{-1}b^{-1}D^{-1}Fbca^{-1}Fc^{-1}b^{-1}cabcFac^{-1}b^{-1}FDbac^2b^{-1}a^{-1}c^{-1} = 1$;

(29) $Dbc^{-1}bDbac^{-1}b^{-1}a^{-1}b^{-1}D^{-1}ca^{-1}b^{-1}D^{-1}Fbcb^{-1}aF = 1$;

(30) $aca^{-1}b^{-1}D^{-1}Fbcb^{-1}aFa^{-1}b^{-1}D^{-1}b^{-1}cac^{-1}Dbac^{-1}b^{-1}cb^{-1}D^{-1} = 1$;

(31) $Dbac^{-1}b^{-1}a^{-1}b^{-1}D^{-1}c^{-1}b^{-1}Fc^{-1}b^{-1}cb^{-1}Fac^{-1}b^{-1}a^{-1}c^{-1}b = 1$;

(32) $D^{-1}aD^{-1}ab^{-1}aFac^{-1}abFac^{-1} = 1$;

(33) $(Fac^{-1}b^{-1}a^{-1}cb^{-1})^2 = 1$;

(34) $(D^{-1}Fac^{-1}abcb^{-1})^2 = 1$;

(35) $(caFab^{-1}a)^4 = 1$;

(36) $dad^{-1} = Dbabca^{-1}Fbc^{-1}a^{-1}b^{-1}D^{-1}$;

(37) $eae^{-1} = babca^{-1}Fbc^{-1}Dba$;

(38) $de\cdot a\cdot e^{-1}d^{-1} = b^{-1}a^{-1}b^{-1}D^{-1}$;

(39) $dbd^{-1} = cb^{-1}Fac^{-1}b^{-1}a^{-1}b^{-1}D^{-1}$;

(40) $ebe^{-1} = Fab^{-1}a$;

(41) $de\cdot b\cdot e^{-1}d^{-1} = c^{-1}b^{-1}Fc^{-1}b^{-1}cb^{-1}Fac^{-1}b^{-1}a^{-1}b^{-1}D^{-1}$;

(42) $dcd^{-1} = DFac^{-1}b^{-1}a^{-1}b^{-1}D^{-1}$;

(43) $ece^{-1} = cb^{-1}D^{-1}cb^{-1}Fac^{-1}b^{-1}a^{-1}b^{-1}$;

(44) $de\cdot c\cdot e^{-1}d^{-1} = Dbc^{-1}b$;

(45) $ede^{-1}d^{-1} = babca^{-1}Fbabca^{-1}Fbc^{-1}b^{-1}cabca^{-1}Fbc^{-1}bcFbc$;

(46) $dede^{-1} = cFbabca^{-1}Fbc^{-1}Dbc^{-1}$;

(47) $e^2 = cb^{-1}D^{-1}cb^{-1}Fac^{-1}b^{-1}a^{-1}b^{-1}Fc^{-1}$;

(48) de^2d^{-1}

$$= c^{-1}b^{-1}Fc^{-1}b^{-1}cb^{-1}Fac^{-1}b^{-1}a^{-1}c^{-1}bcb^{-1}Fac^{-1}b^{-1}a^{-1}b^{-1}Fac^{-1}b^{-1}a^{1}b^{-1}D^{-1} ;$$

(49) $(dbc^{-1})^2 = (db)^4 = (dba)^2 = 1$.

Proof. (1) follows by (vi), (viii), (ix).

(2) follows by (x), (vi), the transformation formulae in Lemma 1 and (xi).

(3)

$$\begin{aligned}(Dbc^{-1}bc)^2 &= (q^2pq^{-1}\cdot q^{-1}p^{-1}q^{-1}(q^{-1}p)^4p^{-1}q^{-1}\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot qp^2q^{-1}p)^2\\ &= (q^2\cdot pqp^{-1}q^{-1}\cdot pqp^{-1}(pq^{-1})^4\cdot q^{-1}\cdot q^2)^2 \quad \text{by (iv), (iii), (vii)}\\ &= 1 \quad \text{by (iv), (vi).}\end{aligned}$$

(4)

$$Dbc^{-1}Dbc^{-1}b^{-1}a^{-1}b^{-1}a^{-1} = q^2pq^2\cdot p^{-1}q^{-1}p^{-1}\cdot(pq^{-1})^4q^2p^{-1}q^{-1}p^2\cdot q^2pq^2\ p^{-1}q^{-1}p^{-1}(pq^{-1})^4\ .$$

$$q^{-1}\cdot q^{-1}pq^{-1}p\cdot pqp^2q^{-1} = q\cdot qpq^2qpq^2\cdot p^{-1}q^2\cdot pq^{-1}pq^{-1}\cdot pq^2p^{-1}\cdot qpqp^2\cdot q^{-1}$$

by (iv), (iii), (vii)

$$= 1 \quad \text{by (iv), (iii), (vii).}$$

(5) Lemma 1 yields

$$p^4dp^{-4} = abcb^{-1}\cdot d\cdot bc^{-1}b^{-1}a^{-1} .$$

Then (i) yields Lemma 2 (5).

(6)

$$\begin{aligned} a^{-1}b^{-1}D^{-1}abc^2b^{-1}a^{-1}Dbac^2 &= p\cdot q^2pqp\cdot(p^{-1}q^{-1})^4\cdot(qp^{-1})^4pqpq^2\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot p^{-1}q \\ &\quad \cdot pq^{-1}p^{-1}qp^2qpq^2p^{-1}q^{-1}p^{-1}(pq^{-1})^4\cdot qpqp\cdot q^{-1}pqpq^{-1}p^{-1}\cdot qpq^{-1}p \\ &= p\cdot q^2pqp^2qp\cdot q^{-1}pq^2pq^{-1}p^{-1}qp^2qpq^2\cdot p^{-1}q^{-1}p^2q^{-1}p^{-1}q^2 \\ &\quad \cdot pqpq^{-1}p^{-1}qpq^{-1}p \quad \text{by (iii), (iv)} \\ &= q^{-1}p^2q^{-1}p^{-1}qpq^2\cdot pq^{-1}pq^{-1}\cdot p^2q^{-1}(qp^2)^8\cdot p^2\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot qpq^{-1}p \\ &\quad \text{by (vii)} \\ &= q^{-1}pq^{-1}\cdot(qpq^{-1}p^{-1})^4\cdot pq^{-1}pq^{-1}p \quad \text{by (ix), (iv), (iii)} \\ &= 1 \quad \text{by (vi), (iv).} \end{aligned}$$

(7)

$$\begin{aligned} D^{-1}abD^{-1}abcb^{-1}cb^{-1} &= pq^2p^{-1}q^2p\cdot q^2p^{-1}q^{-1}p^2q^{-1}p^{-1}\cdot q^2p^{-1}q^2pq^2\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot p^{-1}qp^{-1}q^{-1} \\ &\quad \cdot pqpq^{-1}p^{-1}qp^{-1}q^{-1}p^{-1} \\ &= pq^2p^{-1}q^2\cdot p^2qpq\cdot(q^{-1}p)^4\cdot qpqp\cdot(p^{-1}q^{-1})^4\cdot pqpq^{-1}p^{-1}\cdot qp^{-1}q^{-1}p^{-1} \\ &\quad \text{by (iii), (vii)} \\ &= pq\cdot(qp^{-1})^4q^{-1}p^{-1} \quad \text{by (iii), (iv), (vii)} \\ &= 1 \quad \text{by (iv).} \end{aligned}$$

(8)

$$\begin{aligned} (Dbab^{-1}a^{-1})^2 &= (q^2pq^{-1}\cdot pqpq\cdot(q^{-1}p^{-1})^4\cdot(pq^{-1})^4\cdot qpqp\cdot q^{-1}pqp^2q^{-1})^2 \\ &= (q^2p\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot q^2\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot pqp^2q^{-1})^2 \quad \text{by (iv), (iii), (vii)} \\ &= q^2(p^2q)^8q^{-2} \quad \text{by (iii)} \\ &= 1 \quad \text{by (ix).} \end{aligned}$$

(9)

$$\begin{aligned} (Dbaba^{-1})^2 &= (q^2pq^{-1}\cdot p\cdot qpq\cdot(q^{-1}p^{-1})^4\cdot(pq^{-1})^4\cdot qpqp^2\cdot q^{-1}p^{-1}qp^{-1}q^{-1})^2 \\ &= (q^2p^2\cdot qp\cdot(p^{-1}q^{-1})^4\cdot q^{-1}p^{-1}\cdot qp\cdot(p^{-1}q)^4\cdot q^{-1}p^{-1}\cdot p^2q^2)^2 \quad \text{by (iii), (iv), (vii)} \\ &= 1 \quad \text{by (iii), (iv), (vi).} \end{aligned}$$

(10)

$$Dbacb^{-1}D^{-1}aba^{-1}c^{-1}b^{-1}a^{-1} = q^2pq^{-1}\cdot pqpq\cdot(q^{-1}p^{-1})^4\cdot(pq^{-1})^4\cdot qpqp\cdot q^{-1}\cdot pqpq^2\cdot(qp^{-1})^4$$
$$\cdot pqp\cdot q^2p^{-1}q^{-1}p^2q^{-1}p^{-1}\cdot q^2\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot qpq^{-1}$$
$$= q^2pq^{-1}p\cdot qpq^2pqp^2\cdot qpq^{-1}\cdot pqp^2qpq^2pq^{-1} \quad \text{by (iii), (vii)}$$
$$= q^2p(q^{-1}p^{-1})^4p^{-1}q^2 \quad \text{by (vii)}$$
$$= 1 \quad \text{by (iii).}$$

(11)

$$Dbc^{-1}bDbabca^{-1}b^{-1}D^{-1}b^{-1}cb^{-1}D^{-1}ac^{-1}b^{-1}a^{-1}$$
$$= q^2pq^2\cdot p^{-1}q^{-1}p^{-1}(pq^{-1})^4\cdot q^2p^{-1}q^{-1}p^{-1}\cdot qpq^{-1}p^2q^2pq^{-1}\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot(pq^{-1})^4\cdot qpqp^{-1}q^{-1}p^2q$$
$$\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot(qp^{-1})^4\cdot pqpq^2\cdot p^{-1}q^2p^2\cdot qp^{-1}q^{-1}\cdot pqpq^2\cdot p\cdot(p^{-1}q)^4\cdot qp\cdot q^2p^{-1}q^{-1}pq^{-1}p^2qpq^{-1}$$
$$= q^2pqp^{-1}q^{-1}\cdot p^2q^2p^2q^2\cdot pq^{-1}pqpq^2pqp^{-1}q^{-1}p^2q^2\cdot pqp^2qpq^2\cdot p^{-1}q^2p^2q\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot p^{-1}q^2$$
$$\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot q^{-1}p^{-1}q^{-1}pq^{-1}p^2qpq^{-1} \quad \text{by (iv), (iii), (vii)}$$
$$= q^2p\cdot qp^{-1}qp^{-1}\cdot p^{-1}q^2p^{-1}q^2p^{-1}q^{-1}p^{-1}\cdot(pq)^4\cdot qpqp^{-1}\cdot q^{-1}pq^{-1}p\cdot pqp$$
$$\cdot p(p^2q^2)^4\cdot q^{-1}pq^{-1}\cdot p^{-1}\cdot q^{-1}pq^{-1}p\cdot pqpq^{-1} \quad \text{by (v), (vii), (iii)}$$
$$= q^2p^2q^{-1}pq^{-1}p^{-1}q^2p^{-1}\cdot q^2p^{-1}q^{-1}p^2q^{-1}p^{-1}\cdot q^{-1}pqp^2q^{-1}p\cdot pqpq\cdot p^2q^{-1}p^{-1}q^2$$
$$\text{by (v), (iii), (iv)}$$
$$= q^2p^2q^{-1}pq^{-1}p^{-1}q^{-1}\cdot pq^{-1}\cdot(qp)^4\cdot pq^{-1}pq^{-1}\cdot p^{-1}q^{-1}pq^{-1}p^{-1}q^2 \quad \text{by (iii), (vii)}$$
$$= q^2p(pq^{-1})^4p^{-1}q^2 \quad \text{by (iii), (iv)}$$
$$= 1 \quad \text{by (iv).}$$

(12)

$$abc^{-1}b^{-1}a^{-1}Dbc^{-1}bca^{-1}b^{-1}D^{-1}b^{-1}ca$$
$$= qp^2q^2p^{-1}\cdot(pq)^4\cdot q\cdot p^{-1}q^{-1}p^{-1}(pq^{-1})^4q^2\cdot p^{-1}q^{-1}p^{-1}\cdot qp^2q^{-1}p^2q$$
$$\cdot p^{-1}q^{-1}p^{-1}q^{-1}(qp^{-1})^4\cdot pqpq^2p^{-1}q^2p^2qp^{-1}q^{-1}pqpq^{-1}pqpq^{-1}p^{-1}$$
$$= qp^2q^2p^{-1}\cdot qpqp\cdot q^2\cdot pqpq\cdot p^2q^{-1}p^2\cdot q^2pqp^2qp\cdot q^2p^{-1}q^2p^2qp^{-1}\cdot q^{-1}pqpq^{-1}\cdot pqpq^{-1}p^{-1}$$
$$\text{by (iii), (iv), (vii)}$$
$$= q^{-1}\cdot q^2p^2q^2p^2q^2\cdot qpq\cdot(q^{-1}p)^4\cdot p\cdot q\cdot q^2p^2q^2p^2q^2p^2\cdot p^{-1}\cdot p^{-1}q^{-1}p^{-1}q^{-1}$$
$$\cdot pqpq^{-1}\cdot pqpq^{-1}p^{-1} \quad \text{by (iii), (iv), (vii)}$$
$$= q^{-1}p^2q^2\cdot pq^{-1}pq^{-1}\cdot q^{-1}p^{-1}\cdot qpqp^2qpq\cdot q^2pqpq^{-1}p^{-1} \quad \text{by (iii), (iv), (v)}$$
$$= q^{-1}p^2\cdot q^{-1}p^{-1}q^2(q^{-1}p^{-1})^4p^{-1}q^{-1}p^2q^{-1}p^{-1}(pq)^4\cdot q^2p^{-1} \quad \text{by (iv), (vii)}$$
$$= 1 \quad \text{by (iii), (vii).}$$

(13)

$$D^{-1}abca^{-1}Db^{-1}c^{-1} = pq^2p^{-1}q^{-1}\cdot q^{-1}pq^{-1}p\cdot(p^{-1}q^{-1})^4\cdot(qp^{-1})^4\cdot pq^{-1}p^2q^2p^{-1}q^2pq^2$$
$$pqp^{-1}q^{-1}p\cdot pq\cdot p^{-1}q^{-1}p^{-1}q^2p^{-1}q^{-1}p^{-1}\cdot pqpq^2p^{-1}$$
$$= pq^2p^{-1}q^{-1}p^{-1}qp\cdot p^2qpq^{-1}\cdot p^2qpq^{-1}\cdot(qp^{-1})^4\cdot pq^{-1}\cdot pqpq^2pqp$$
$$p^2q^{-1}pq^{-1}p^{-1}(pq)^4\cdot qpqpqp\cdot p^{-1}q^{-1}pqpq^2p^{-1} \quad \text{by (iii), (iv), (vii)}$$
$$= pq^2p^{-1}q^{-1}p^{-1}qp\cdot(p^2qpq^{-1})^4\cdot p^{-1}q^{-1}pqpq^2p^{-1} \quad \text{by (iii), (iv), (vii).}$$

Then (xii) yields Lemma 2 (13).

(14)

$$Dba^2b = q^2p\cdot q^2p^{-1}q^{-1}p^2\cdot p(pq^{-1})^4\cdot qpqp\cdot q^{-1}p^{-1}qp^2q^{-1}p^2$$
$$= q^2p^2qp^2q^2p^2q^{-1}p^2 \quad \text{by (iii), (iv), (vii)}$$
$$= q^2p^2q^{-1}p^2\cdot q^2p^2qp^2 \quad \text{by (v).}$$

By (v), (xiii) follows Lemma 2 (14).

(15)

$$Dba^2bc^{-1}bc^{-1}b = q^2p\cdot q^2p^{-1}q^{-1}p^2q^{-1}p^{-1}q^{-1}\cdot(qp)^2\cdot(pq^{-1})^2\cdot(qp)^2\cdot q^{-1}p^{-1}qp^2q^{-1}pq$$
$$p^{-1}q^{-1}p^{-1}qpq^{-1}pqp^{-1}q^{-1}p^{-1}qpq^{-1}p^2$$
$$= q^2p^2\cdot q^{-1}\cdot q^2p^2q^2p^2q^2p^2\cdot p\cdot(pq)^4\cdot q^{-1}p^{-1}q^{-1}p^2q^{-1}p^{-1}q^{-1}\cdot q^2pq^{-1}$$
$$pqp^{-1}q^{-1}p^{-1}qpq^{-1}p^2 \quad \text{by (iii), (iv), (vii)}$$
$$= q^2p^2q^{-1}p^2q^2\cdot pqpq\cdot p^2qp\cdot q^{-1}pq^{-1}p\cdot qp^{-1}q^{-1}p^{-1}qpq^{-1}p^2 \quad \text{by (iii), (v), (vii)}$$
$$= q^2p^2q^{-1}p^2qp^{-1}\cdot q^{-1}pq^{-1}p\cdot p^{-1}q^{-1}p^{-1}q^2p^{-1}q^{-1}p^{-1}\cdot qpq^{-1}p^2 \quad \text{by (iii), (iv)}$$
$$= q^2p^2q^{-1}\cdot p^2qp^2qp^{-1}\cdot qpq\cdot p\cdot q^2\cdot pqpq\cdot pq^{-1}p^2 \quad \text{by (iv), (vii)}$$
$$= q^2p^2q^{-1}\cdot p^2qp^2qp^2q^{-1}\cdot p^2q^2p^2q^2p^2q^{-1}\cdot qp^2q^2 \quad \text{by (iii)}$$
$$= q^2p^2q^{-1}\cdot(p^2q)^4qp^2q^2 \quad \text{by (v)}$$
$$= 1 \quad \text{by (ix).}$$

(16) and (17)

$$(c^{-1}b^{-1}a^{-1}Dca^{-1})^2 = (p^{-1}\cdot p^{-1}q^{-1}\cdot p^{-1}(pq)^4\cdot qp^{-1}qp^{-1}\cdot pqpq^2pqp\cdot q^{-1}p^2qp^{-1}q^{-1})^2$$
$$= (p^2q^2p\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot q^2\cdot qp(p^{-1}q^{-1})^4\cdot qp^{-1}qp^{-1}\cdot q^{-1})^2$$
$$\text{by (iii), (iv), (vii)}$$
$$= p^2q^2p^2qp^{-1}q^{-1}p\cdot q^2p^2q^2p^2\cdot qp^{-1}q^{-1}pq^2 \quad \text{by (iii), (iv)}$$
$$= p^2q^2p^2q^2\cdot q^{-1}p^{-1}q^{-1}p^{-1}q^{-1}\cdot q^{-1}p^{-1}\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot pq^2 \quad \text{by (v)}$$
$$= q^2p^2q^{-1}\cdot q^{-1}p^{-1}qp\cdot q^{-1}p^{-1}qp\ qp^2q^2 \quad \text{by (iii), (v).}$$

Then follows Lemma 2 (16) by (vi) and moreover

$$
\begin{aligned}
&(c^{-1}b^{-1}a^{-1}Dca^{-1})^2 \cdot b^2 \\
&= q^2p^2q^{-1} \cdot q^{-1}p^{-1}qpq^{-1}p^{-1}qp \cdot p^{-1} \cdot q^2p^2q^2p^2q^2p^2p^{-1} \cdot p^{-1}q^{-1}p^{-1}q^{-1} \cdot pqp^{-1}q^{-1}p^{-1} \\
&= q^2p^{-1} \cdot p^{-1}q^{-1}p^{-1}q^{-1} \cdot pq \cdot p^{-1}q^{-1}p^{-1}q^{-1} \cdot q^{-1}p^{-1} \cdot qpqp^2qpqq^{-1}p^2q^{-1}p^{-1} \quad \text{by (iii), (v), (vi)} \\
&= q^2p^{-1}q \cdot pqp^2q^2 \cdot pqp^2q \cdot p^{-1}q^{-1}p^{-1}q^{-1} \cdot q^{-1}p^2q^{-1}p^{-1} \quad \text{by (iii), (vii)} \\
&= q^2p^{-1}q \cdot pqp^2q^2 \cdot pqp^2q^2 \cdot pqp^2q^2 \cdot q^2p^{-1}q^{-1}p^2q^{-1}p^{-1} \quad \text{by (iii)} \\
&= q^2p^{-1}q \cdot (pqp^2q^2)^4 \cdot q^{-1}pq^2 \quad \text{by (vii).}
\end{aligned}
$$

Then follows Lemma 2 (17) by (xiv).

(18)

$$
\begin{aligned}
DbaDbc^{-1}bDbac^{-1} &= q^2p \cdot q^2p^{-1}q^{-1} \cdot (q^{-1}p)^4p^{-1} \cdot qpqp \cdot q^{-1}p^{-1}q^2pq^2 \cdot p^{-1}q^{-1}p^{-1} \cdot (pq^{-1})^4 \\
&\qquad q^2p^{-1}q^{-1}p^{-1} \cdot qp \cdot q^{-1}p^2q^2p \cdot q^2p^{-1}q^{-1}p^{-1} \cdot (pq^{-1})^4 \cdot qpqp \cdot q^{-1}p^2qp^{-1}q^{-1}p^2 \\
&= q^2p^2qp^2q^{-1}pqp^{-1}q^{-1} \cdot p^2q^2p^2q^2 \cdot p^2qp^2 \cdot qp^{-1}qp^{-1} \cdot q^{-1}p^2 \\
&\qquad \text{by (iii), (iv), (vii)} \\
&= q^2p^2q^2 \cdot q^{-1}p^2 \cdot q^{-1}p^2 \cdot (p^{-1}q)^4 \cdot q^{-1}pq^{-1} \cdot qp(p^{-1}q^{-1})^4 \cdot q \cdot p^2q^2p^2q^2 \cdot q^2 \\
&= q^2p^2q^2 \cdot (q^{-1}p^2)^4 \cdot q^2p^2q^2 \quad \text{by (iii), (iv), (v).}
\end{aligned}
$$

Then follows Lemma 2 (18) by (ix).

(19)

$$
Fa^2 = pq^2pq \cdot qpq^{-1}p^{-1}(pq^{-1})^4qpq^{-1}p^{-1}(pq)^4 \cdot q^{-1}p^{-1}q^2p^{-1} .
$$

Then (iii), (iv), (vi) obviously yield Lemma 2 (19).

(20)

$$
\begin{aligned}
b^{-1}Fbc^2b^{-1}Fbc^2 &= p^2qp^{-1} \cdot qpq^2pqp^2 \cdot p^2 \cdot qpq^2pqp^2 \cdot q^{-1}p^{-1}qpq^{-1} \cdot p^{-1}qp^{-1}q \\
&\qquad pq^2pq^2pq^2pqp^2q^{-1}p^{-1}qpq^{-1}p \\
&= p^2 \cdot qpqp \cdot (p^{-1}q^2)^8 \cdot q^2pq^2 \cdot pq^{-1}pq^{-1} \cdot q^{-1}pq^{-1}p \cdot pq^2pq^2pq^2 \\
&\qquad pqp^2q^{-1}p^{-1}qpq^{-1}p \quad \text{by (vii), (iv)} \\
&= p \cdot q^{-1}p^{-1}qpq^{-1}p^{-1}qp \cdot pqp^{-1}q \cdot pq^2pq^2pq^2pq^2 \cdot q^{-1}p^2q^{-1} \\
&\qquad p^{-1}qpq^{-1}p \quad \text{by (ix), (iv), (iii)} \\
&= q^{-1}pqp^{-1}q^{-1} \cdot pqpq \cdot p^{-1}q^{-1}p^{-1}q^2p^{-1}q^2p^{-1} \cdot q^2p^{-1}q^{-1}p^2q^{-1}p^{-1} \cdot qpq^{-1}p \\
&\qquad \text{by (ix), (vi)} \\
&= q^{-1}pqp^{-1}q^2p^{-1}q^{-1} \cdot p^2q^{-1}p^{-1}q^2p^{-1}q^{-1} \cdot p^2q \cdot pq^{-1}pq^{-1} \cdot p \quad \text{by (vii), (iii)} \\
&= 1 \quad \text{by (vii), (iv).}
\end{aligned}
$$

(21)

$$a^{-1}Fa^{-1}b^2c^2 = p^{-1}\cdot p^{-1}q^{-1}(qp^{-1})^4pq\cdot p^{-1}q^{-1}pq(q^{-1}p^{-1})^4p^2qpq^{-1}p^{-1}qp^2q^{-1}p^{-1}qpq^{-1}p$$
$$= p^{-1}q^{-1}p^2\cdot pqpq^2p\cdot(p^{-1}q)^4\cdot qp\cdot p^{-1}q^2p^2q^{-1}p^{-1}qp^2q^{-1}p^{-1}qpq^{-1}p \quad \text{by (iv), (iii)}$$
$$= p^{-1}q^{-1}pq^{-1}p^{-1}q\cdot qp^{-1}qp^{-1}\cdot p\cdot q^2p^2q^2p^2q^2p\cdot p^{-1}qp^{-1}q\cdot p^2q^{-1}\cdot p^{-1}qpq^{-1}p \quad \text{by (iv), (vii)}$$
$$= p^{-1}q^{-1}pq^{-1}p^{-1}\cdot qpq^{-1}pq^{-1}p^{-1}q^2p^{-1}q^{-1}p^2\cdot p^{-1}q^{-1}p^{-1}q^{-1}p^{-1}\cdot qpq^{-1}p \quad \text{by (v), (iv)}$$
$$= p^{-1}q^{-1}pq^{-1}p^{-1}qp\cdot q^{-1}p^{-1}q^{-1}p^{-1}(pq^2)^8\cdot q^2p^{-1}q^2p^{-1}q^2p^{-1}qp \quad \text{by (vii), (iii)}$$
$$= p^{-1}q^{-1}pq^{-1}p^{-1}qp^2\cdot qpq^{-1}p^{-1}\cdot qp\cdot(p^{-1}q)^4\cdot q^{-1}p^{-1}\cdot p^2q^{-1}pqp^{-1}qp \quad \text{by (ix), (iii).}$$

Then by (iv), (vi) we have $(a^{-1}Fa^{-1}b^2c^2)^2 = 1$. Hence Lemma 2 (19) yields Lemma 2 (21).

(22)

$$Fab^{-1}ab^{-1}Fbc^2abc^2a = pq^2pq^2\cdot p^2\cdot p^{-1}q^{-1}p^{-1}\cdot(pq^{-1})^4\cdot q^2p^{-1}q^{-1}p^{-1}\cdot qpq^{-1}pqp^{-1}\cdot qpq^2pqp^2\cdot p^2$$
$$\cdot qpq^2pqp^2\cdot q^{-1}p^{-1}qpq^{-1}pqp^{-1}q^{-1}p^{-1}qpq^{-1}pqpq^{-1}p^{-1}$$
$$= pq^2pq^2p^{-1}qpq\cdot(qp)^4\cdot p^{-1}q\cdot(qp)^4\cdot p^{-1}q^{-1}\cdot p^{-1}q^{-2}p^{-1}q^2p^{-1}q^2p^{-1}q^2p^{-1}q^2$$
$$\cdot p^{-1}qpq^{-1}pqp^{-1}q^{-1}p^{-1}qpq^{-1}pqpq^{-1}p^{-1} \quad \text{by (iv), (vii)}$$
$$= pq^2pq^2p^{-1}qpq\cdot p^{-1}qp^{-1}q\cdot pq^{-1}\cdot q^{-1}pq^{-1}p\cdot q^{-1}pqp^{-1}q^{-1}p^{-1}qpq^{-1}pqpq^{-1}p^{-1} \quad \text{by (iii), (ix)}$$
$$= pq^2pq\cdot qp^{-1}qp^{-1}\cdot p^{-1}q^{-1}\cdot p^2q^{-1}p^{-1}q^2p^{-1}q^{-1}\cdot p^2q^{-1}p^{-1}\cdot pqpqp$$
$$\cdot q^{-1}pqpq^{-1}p^{-1} \quad \text{by (iv)}$$
$$= pq(pq)^4q^{-1}p^{-1} \quad \text{by (vii), (iv), (iii)}$$
$$= 1 \quad \text{by (iii).}$$

(23)

$$Fab^{-1}abca^{-1}Fbc^{-1}a^{-1}c^{-1}b^{-1}a^{-1}ca$$
$$= pq^2pq^2pq^{-1}p^{-1}\cdot(pq^{-1})^4\cdot q^2p^{-1}q^{-1}p^2\cdot pqp^{-1}q^{-1}\cdot p^2\cdot qp^{-1}qp^{-1}q^2p^{-1}q^2p^{-1}\cdot q^{-1}pq^{-1}p$$
$$\cdot qp^{-1}q^{-1}p^{-1}qp^{-1}q^{-1}p^{-1}qpq^{-1}p^2qpq^{-1}pqpq^{-1}p^{-1}$$
$$= pq^2pq^2p^{-1}qpq^{-1}pqp^2\cdot qpqpqp^{-1}\cdot q^2p^2q^2p^2\cdot qpq^2\cdot pqpq$$
$$p^{-1}q^{-1}p^{-1}qpq^{-1}p^2qpq^{-1}pqpq^{-1}p^{-1} \quad \text{by (iv), (vii), (iii)}$$
$$= pq^2pq^2p^{-1}qpq^{-1}\cdot pqpq\cdot p^2q^2\cdot p^2q^{-1}p^{-1}q^2p^{-1}q^{-1}\cdot p^{-1}qp^{-1}q^{-1}p^{-1}$$
$$qpq^{-1}p^2qpq^{-1}pqpq^{-1}p^{-1} \quad \text{by (v), (iii)}$$

$$= pq^2pq^2p^{-1}qpq^2p^2qp^{-1}(pq^{-1})^4\cdot q^{-1}\cdot q^{-1}p^{-1}q^{-1}(qp)^4\cdot p^2q^{-1}p^{-1}$$
$$qpq^{-1}p^2qpq^{-1}pqpq^{-1}p^{-1} \quad \text{by (vii), (iii)}$$
$$= pq^2pq\cdot qp^{-1}qp^{-1}\cdot p^2q^2p^2q^2p^2q^2(q^{-1}p)^4p^{-1}\cdot qpqp\cdot q^{-1}pqpq^{-1}p^{-1}$$
$$\text{by (iv), (iii), (vii)}$$
$$= pq(qp)^4q^{-1}p^{-1} = 1 \quad \text{by (v), (iv), (iii).}$$

(24)

$$abca^{-1}Fbc^{-1} = q\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot(qp^{-1})^4pqp^{-1}q^2p^2q\cdot(q^{-1}p)^4q\cdot q^{-1}p^{-1}q^2p^{-1}q^{-1}p^2$$
$$= q^2pqp^2qp\cdot p^2q^2p^2q^2\cdot p^2qpq^2pq \quad \text{by (iv), (iii), (vii)}$$
$$= p^{-1}q^{-1}pq\cdot(q^{-1}p)^4p^{-1}q^{-1}pq \quad \text{by (v), (vii).}$$

By (vi), (iv) we have then $(abca^{-1}Fbc^{-1})^2 = 1$. Moreover

$$abca^{-1}Fbc^{-1}\cdot b^2\cdot abca^{-1}Fbc^{-1}b^2 = p^{-1}q^{-1}pqp^{-1}q^{-1}\cdot pqpq\cdot pq^{-1}p^{-1}qp\cdot q^{-1}pq^{-1}p\cdot qp^{-1}q^{-1}\cdot pqpq$$
$$\cdot pq^{-1}p^{-1}qpq^{-1}p^2$$
$$= p^{-1}\cdot q^{-1}pq^{-1}p\cdot(p^{-1}q^2)^8\cdot q^{-1}pq^{-1}p\cdot p \quad \text{by (iii), (iv)}$$
$$= 1 \quad \text{by (ix), (iv).}$$

(25)

$$c^{-1}bcbcFac^{-1}b^{-1}FDbac^2b^{-1}a^{-1}c^{-1}Fac^{-1}b^{-1}FDbac^2b^{-1}a^{-1}$$
$$= p^{-1}qp^{-1}q^{-1}p^{-1}qp^2q^{-1}p^2qp^2q^{-1}p^2q^2pq^2pq^2\cdot pq^{-1}pq^{-1}\cdot p\cdot pqp^2qpq^2\cdot pq^2p^2q^2p^{-1}q^2\cdot pq^{-1}pq^{-1}$$
$$\cdot p^2qpq^{-1}p\cdot qpq^{-1}p^{-1}qpq^{-1}p^{-1}\cdot qp^2q^{-1}p^{-1}q\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot q^{-1}pq^2pq^2\cdot pq^{-1}pq^{-1}\cdot p\cdot pqp^2qpq^2$$
$$\cdot pq^2p^2q^2p^{-1}q^2\cdot pq^{-1}pq^{-1}\cdot p^2qpq^{-1}p\cdot qpq^{-1}p^{-1}\cdot qpq^{-1}p^{-1}\cdot qp^2q^{-1}$$
$$= p^{-1}qp^{-1}q^{-1}p^{-1}qp^2q^{-1}p^2qp^2q^{-1}p^2q^2pq^2p\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot q^{-1}p^2qp^2q^{-1}\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot qpqp$$
$$\cdot q^{-1}p^2qp^{-1}q^{-1}pqpq^{-1}p^{-1}qp^{-1}q^{-1}p^{-1}q^2pq^2p\cdot q^{-1}p^{-1}q^{-1}p^{-1}q^{-1}p^2q\cdot p^2q^{-1}\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot qpqp$$
$$\cdot q^{-1}p^2qp^{-1}q^{-1}pqpq^{-1} \quad \text{by (iii), (iv), (vi), (vii)}$$
$$= p^{-1}qp^{-1}q^{-1}p^{-1}qp^2q^{-1}p^2qp^2q^{-1}p^2q^2pq^2p^{-1}q^{-1}\cdot p^{-1}qp^{-1}q\cdot p^{-1}q^{-1}pqpq^{-1}$$
$$\cdot p^{-1}qp^{-1}q^{-1}p^{-1}q^2pq^2p^2q\cdot p^{-1}qp^{-1}q\cdot p^2\cdot qp^{-1}qp^{-1}\cdot q^{-1}pqpq^{-1} \quad \text{by (iii), (iv), (vii)}$$
$$= p^{-1}qp^{-1}q^{-1}p^{-1}qp^2q^{-1}p^2qp^2q^{-1}p^2q^2pq\cdot qp^{-1}qp^{-1}\cdot pqpq^2pqp\cdot q^{-1}p^{-1}\cdot q\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot q^{-1}p$$
$$\cdot q^2p^{-1}q^2pq^2pqpq^{-1} \quad \text{by (iii), (iv), (vii)}$$
$$= p^{-1}qp^{-1}q^{-1}p^{-1}qp^2q^{-1}p^2q\cdot p^2q^{-1}p^2q^2pqpq^{-1}p^2q\cdot p^2q^{-1}pqpq^{-1}pqpq^{-1}\cdot p^2qp^2qp$$
$$\text{by (iii), (iv), (vii)}$$
$$= p^{-1}qp^{-1}q^{-1}p^{-1}q\cdot p^2q^{-1}p^2qpqp^{-1}qpq^{-1}p^{-1}q^{-1}pqp^{-1}qp^2\cdot q^{-1}pqpq^{-1}pqp\cdot q^{-1}p^2qp^2qp$$
$$\text{by (iii), (iv), (vii)}$$

$= p^{-1}qp^{-1}q^{-1}p^{-1}qpqp^{-1}qpqp^{-1}q^2p\cdot q^2p^2q^{-1}p^{-1}q^{-1}pqpq^{-1}\cdot pqpq^{-1}p^2qp^2qp$ by (iii), (vii)

$= p^{-1}qp^{-1}q^{-1}p^{-1}qpqp^{-1}qpqp^{-1}\cdot q^2pq^{-1}pqp^{-1}q^{-1}p^{-1}qpqpq^{-1}p^2qp^2\cdot qp$ by (iii), (vii)

$= p^{-1}qp^{-1}q^{-1}p^{-1}qpq\cdot p^{-1}qpqp^{-1}q^{-1}p^{-1}q^{-1}q^{-1}p^{-1}q^{-1}p\cdot q^{-1}p^{-1}qp$ by (iii), (vii)

$= p^{-1}qp^{-1}q^{-1}p^{-1}\cdot qpqp^2qpq\cdot p^{-1}q^{-1}p^{-1}qp$ by (iii), (vii)

$= p^{-1}q^2\left(q^{-1}p^{-1}\right)^4\left(p^{-1}q^{-1}\right)^4q^2p$ by (iii), (vii)

$= 1$ by (iii).

(26)

$caFac^{-1}b^{-1}FDbac^2b^{-1}Fac^{-1}b^{-1}FDba^2bc$

$= p^2qpq^{-1}\cdot pqpq\cdot pq^2pq^2\cdot pq^{-1}pq^{-1}\cdot p\cdot pqp^2qpq^2\cdot pq^2p^2q^2p^{-1}q^2pq^{-1}pq^{-1}\cdot p^2qpq^{-1}p\cdot qpq^{-1}p^{-1}$

$\cdot qpq^{-1}p^{-1}\cdot qp^{-1}qpq^2pq^2\cdot pq^{-1}pq^{-1}\cdot p\cdot pqp^2qpq^2\cdot pq^2p^2q^2p^{-1}q^2\cdot pq^{-1}pq^{-1}\cdot p^2qpq^{-1}p^{-1}qp^{-1}q^{-1}p$

$= p^2qpq^2p^{-1}qp\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot q^{-1}p^2qp^2q^{-1}\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot qpqp\cdot q^{-1}p^2\cdot qp^{-1}q^{-1}\cdot pqp^2qpq^2$

$\cdot p\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot q^{-1}p^2qp^2q^{-1}\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot qpqp\cdot q^{-1}p^{-1}qp^{-1}q^{-1}p$

by (iii), (iv), (vi), (vii)

$= p^2qpq^2p^{-1}qp^{-1}\cdot p^{-1}qp^{-1}q\cdot p^2\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot q^2\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot p^2\cdot qp^{-1}qp^{-1}\cdot q^{-1}p^2q^{-1}p$

$\cdot qp^{-1}qp^{-1}\cdot p^{-1}\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot q^2\cdot p^{-1}q^{-1}p^{-1}q^{-1}p^{-1}qp^{-1}q^{-1}p$ by (iii), (iv), (vii)

$= p^2qpq^2\cdot p^{-1}qp^{-1}q\cdot p^2q^{-1}\cdot p^2q^{-1}p^{-1}q^2p^{-1}q^{-1}\cdot p$ by (iii), (iv)

$= 1$ by (iii), (iv), (vii).

(27)

$aca^{-1}b^{-1}D^{-1}Fbca^{-1}Fa^{-1}c^{-1}Fac^{-1}b^{-1}FDba$

$= qpq^{-1}pqpq^{-1}p^2qp^{-1}q^{-1}p^2\cdot qp^{-1}qp^{-1}\cdot q^2pq^2p^2q^{-1}\cdot q^{-1}p^{-1}q^2p^{-1}q^{-1}p^2\cdot q^{-1}p^2\cdot qp^{-1}qp^{-1}\cdot q^2p^{-1}q^2$

$\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot p^{-1}q\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot q^{-1}pq^2pq^2\cdot pq^{-1}pq^{-1}\cdot p\cdot pqp^2qpq^2\cdot pq^2p^2q^2p^{-1}q^2$

$\cdot pq^{-1}pq^{-1}\cdot p^2qpq^{-1}p^{-1}$

$= qpq^{-1}pqpq^{-1}p^2q\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot pqpq^2p^2q^{-1}p^2\cdot qpqp\cdot qp^{-1}\cdot q^{-1}pq^{-1}p\cdot qpq^{-1}pq^2p\cdot q^{-1}p^{-1}q^{-1}p^{-1}$

$\cdot q^{-1}p^2qp^2q^{-1}\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot qpqp\cdot q^{-1}p^{-1}$ by (iii), (iv), (vii)

$= qpq^{-1}pqpq^{-1}p^2\cdot q^2pqp^2qp\cdot q^2p^2\cdot q^{-1}pq^{-1}p\cdot qp^{-1}q^{-1}\cdot q^{-1}pq^{-1}p\cdot q^2p^{-1}\cdot p^{-1}qp^{-1}q\cdot p^2\cdot q^{-1}p^{-1}q^{-1}p^{-1}$

$\cdot q^2\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot p^{-1}$ by (iii), (iv), (vii)

$= qpq^{-1}pq\cdot pq^{-1}pq^{-1}pq^{-1}\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot q\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot p^{-1}q$ by (iii), (iv), (vii)

$= q\left(pq^{-1}\right)^4q^{-1} = 1$ by (iii), (iv).

(28)

$bc^2a^{-1}b^{-1}D^{-1}Fbca^{-1}Fc^{-1}b^{-1}cabcFac^{-1}b^{-1}FDbac^2b^{-1}a^{-1}c^{-1}$

$= pqp^2q^{-1}p^{-1}qpq^{-1}p^2q\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot(qp^{-1})^4\cdot pqpq^2p^2q^{-1}\cdot q^{-1}p^{-1}q^2p^{-1}q^{-1}p^2\cdot q^{-1}p^2\cdot qp^{-1}qp^{-1}$

$\cdot q^2p^{-1}q^2p^{-1}q^2p^2qp^2q^{-1}pqpq^{-1}pqp^{-1}q^{-1}p^2q^2pq^2pq^2\cdot pq^{-1}pq^{-1}\cdot p\cdot pqp^2qpq^2\cdot pq^2p^2q^2p^{-1}q^2$

$\cdot pq^{-1}pq^{-1}\cdot p^2qpq^{-1}p\cdot qpq^{-1}p^{-1}qpq^{-1}p^{-1}\cdot qp^2q^{-1}p^{-1}qp^{-1}q^{-1}p^2$

$= pqp^2q^{-1}p^{-1}qpq^{-1}p^2q\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot pqpq^2p^2q^{-1}p^2\cdot qpqp\cdot qp^{-1}q^2p^{-1}q^2p^2qp^2q^{-1}pqpq^{-1}pqp^{-1}q^{-1}$

$p^2q^2pq^2p\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot q^{-1}p^2qp^2q^2p^{-1}q^{-1}p^{-1}\cdot qpqp\cdot q^{-1}p^2qp^{-1}q^{-1}pqpq^{-1}p^{-1}qp^{-1}q^{-1}p^2$

by (iv), (vi), (vii)

$= pqp^2q^{-1}p^{-1}qpq^{-1}p^2\cdot q^2pqp^2qp\cdot q^2p\cdot pq^{-1}pq^{-1}\cdot p^2q^2p^{-1}q^2p^2qp^2q^{-1}pqpq^{-1}pqp^{-1}q^{-1}p^2q^2pq^2p^{-1}$

$\cdot p^{-1}qp^{-1}q\cdot p^2\cdot q^2p^{-1}q^{-1}p^2q^{-1}p^{-1}\cdot q^2p^2qp^{-1}q^{-1}pqpq^{-1}p^{-1}qp^{-1}q^{-1}p^2$ by (iii)

$= p\cdot qp^2q^{-1}p^{-1}q^2p^{-1}\cdot(pq^{-1})^4\cdot q^{-1}\cdot pq^2p^{-1}q^2p^2qp^2q^{-1}pqpq^{-1}pqp^{-1}q^{-1}p^2q^2p\cdot q^2p^{-1}q^2\cdot(qp^{-1})^4$

$\cdot pq^2pqpq^{-1}p^{-1}q\cdot p^{-1}q^{-1}p^2$ by (iv), (vii)

$= pq^2pq\cdot qpqp\cdot pq^{-1}pq^{-1}\cdot q^{-1}p^{-1}q^2p^2qp^2q^{-1}pqpq^{-1}pqp^{-1}q^{-1}\cdot p^2q^2pqpq^{-1}p^2qp^2q^2p^{-1}q^{-1}p^2$

by (iii), (iv), (vi)

$= pq^2pqp^{-1}q^{-1}p^2qp^{-1}q^{-1}p^{-1}q^2p^2qp^2q^{-1}\cdot pqpq^{-1}pqp^{-1}\cdot q^{-1}pq^{-1}p^{-1}qpqp^{-1}q^{-1}p^{-1}qpq^2pq$

by (iii), (iv), (vii)

$= q^{-1}p^{-1}q\cdot pq^2p^{-1}q^2pq^2p^2q^{-1}p^{-1}q^{-1}pqpq^{-1}pqp^{-1}\cdot q^{-1}pq^{-1}p^{-1}qpqp^{-1}q^{-1}p^{-1}\cdot qpq^{-1}\cdot q^{-1}pq$

by (iii), (iv), (vii)

$= q^{-1}p^{-1}q\cdot pq^{-1}pq\cdot pq^{-1}pq^{-1}\cdot p^{-1}\cdot q^{-1}pq^{-1}p\cdot q^{-1}p^{-1}qpqp^{-1}q^{-1}p^{-1}qpqq^{-1}pq$

by (iii), (iv), (vii)

$= q^{-1}p^{-1}q\cdot pq^{-1}pq^{-1}\cdot q^{-1}p^{-1}\cdot qpqp^2qpq\cdot p^{-1}q^{-1}p^{-1}qpq^{-1}\cdot q^{-1}pq$ by (iv)

$= q^{-1}p^{-1}q\cdot(pq^{-1})^4\cdot q^{-1}pq = 1$ by (iii), (iv), (vii).

(29)

$Dbc^{-1}bDbac^{-1}b^{-1}a^{-1}b^{-1}D^{-1}ca^{-1}b^{-1}D^{-1}Fbcb^{-1}aF$

$= q^2pq^2\cdot p^{-1}q^2\cdot pq^{-1}\cdot pq^{-1}\cdot pqp^{-1}q^{-1}p^{-1}\cdot qpq^{-1}p^2q^2pq^2p^{-1}q^2\cdot pq^{-1}pq^{-1}\cdot p^2qpq^{-1}p^2qpq^{-1}p^2$

$\cdot qp^{-1}qp^{-1}\cdot q^2pq^2p^{-1}q^2p^2qpq^{-1}p^2qp^{-1}q^{-1}p^2\cdot qp^{-1}qp^{-1}\cdot q^2pq^2p^2q^2p^{-1}\cdot q^2p^{-1}q^{-1}p^2q^{-1}p^{-1}$

$\cdot qp^{-1}q^{-1}p^{-1}\cdot qpqp\cdot q^2pq^2pq^2$

$= q^2pq^2\cdot pqpq\cdot qpqp\cdot qpq^{-1}p^2q^2pq^2p^{-1}q^{-1}p^{-1}\cdot qpqp\cdot q^{-1}p^2qpq^{-1}p^{-1}q^{-1}\cdot pqpq\cdot qp^{-1}q^2p^2qpq^{-1}p^2q$

$\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot pqpq^2p^2q^{-1}p^2qp\cdot q^{-1}p^{-1}q^{-1}p^2q^{-1}p^{-1}q^{-1}\cdot q^2pq^2pq^2$ by (iii), (iv), (vii)

$= q^2pqp^{-1}q^{-1}\cdot p^2q^2p^2q^2\cdot p\cdot q^2p^{-1}q^{-1}p^2q^{-1}p^{-1}\cdot q^2p^2qp^2\cdot p^{-1}q^{-1}p^{-1}q^2p^{-1}q^{-1}p^{-1}\cdot qp^{-1}q^2p^2qpq^{-1}p^2$

$\cdot q^2pqp^2qp\cdot q^2p^2q^{-1}p^2\cdot qpqp\cdot qp^2qp\cdot q^{-1}pq^2pq^2$ by (iii), (iv), (vii)

$= q^2pqp^{-1}qp^2q^{-1}p^2qp^{-1}qp^{-1}\cdot qpq^2\cdot pqpq\cdot p^{-1}q^2p^2q\cdot pq^{-1}pq^{-1}\cdot p^2\cdot q^{-1}pq^{-1}p\cdot q^{-1}pqpq^{-1}pq^2pq^2$ by (v), (iii), (vii), (iv)

$= q^2p\cdot qp^{-1}qp^{-1}\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot p^2qp^{-1}q^{-1}\cdot p^2q^2p^2q^2\cdot p^{-1}\cdot q^{-1}pq^{-1}p\cdot q^2pq^2$ by (iii), (iv)

$= q^2p^2q^{-1}p^2\cdot qp^{-1}qp^{-1}\cdot qp^{-1}\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot pq^2 = 1$ by (v), (iii), (iv).

(30)

$aca^{-1}b^{-1}D^{-1}Fbcb^{-1}aFa^{-1}b^{-1}D^{-1}b^{-1}cac^{-1}Dbac^{-1}b^{-1}cb^{-1}D^{-1}$

$= qpq^{-1}pqpq^{-1}p^2qp^{-1}q^{-1}p^2\cdot qp^{-1}qp^{-1}\cdot q^2pq^2p^2q^2\cdot p^{-1}q^2p^{-1}q^{-1}p^2q^{-1}\cdot p^{-1}qp^{-1}q^{-1}p^{-1}\cdot qpqp$
$\cdot q^2pq^2\cdot pq^2pqp^{-1}q^{-1}p^2\cdot qp^{-1}qp^{-1}\cdot q^2pq^2p^{-1}q^2p^2qp^{-1}q^{-1}pqpq^{-1}pqpq^{-1}p^2qp^{-1}q^{-1}p^2q^2pq^2p^{-1}q^2$
$\cdot pq^{-1}pq^{-1}\cdot p^2qpq^{-1}p^2qp^2q^{-1}pqpq^{-1}\cdot p^{-1}qp^{-1}q\cdot p^{-1}q^2pq^2p^{-1}q^2$

$= qpq^{-1}pqpq^{-1}p^2q\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot pqpq^2p^2q^{-1}p^2qp\cdot q^{-1}p^{-1}q^{-1}p^2q^{-1}p^{-1}q^{-1}\cdot q^2pq^2pq^2pq$
$\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot pqpq^2p^{-1}q^2p^2qp^{-1}q^{-1}pqpq^{-1}pqpq^{-1}\cdot p^2qp^{-1}q^{-1}p^2q^2pq^2p^{-1}q^{-1}p^{-1}$
$\cdot qpqp\cdot q^{-1}p^2qp^2q^{-1}\cdot pqpq^2pqp\cdot q^2p^{-1}q^2$

$= qpq^{-1}pqpq^{-1}p^2\cdot q^2pqp^2qp\cdot q^2p^2q^{-1}p^2\cdot qpqp\cdot qp^2qpq^{-1}pq^2pq^2p\cdot q^2pqp^2qp\cdot q^2p^{-1}q^2p^2qp^{-1}q^{-1}$
$pqpq^{-1}pqpq^{-1}p^2qp^{-1}q^{-1}p^2q^2p\cdot q^2p^{-1}q^{-1}p^2q^{-1}p^{-1}\cdot q^2p^2qp^2\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot q^2\cdot p^{-1}q^{-1}p^{-1}q^{-1}$
$\cdot q^{-1}p^{-1}q^2$ by (iii), (vi), (vii)

$= qpq^{-1}pq\cdot pq^{-1}pq^{-1}\cdot p^2\cdot q^{-1}pq^{-1}p\cdot q^{-1}pqpq^{-1}pq^2pqp^2\cdot q^{-1}p^2q^2p^2q\cdot p^{-1}q^{-1}pqpq^{-1}pqpq^{-1}p^2qp^{-1}$
$\cdot q^{-1}p^2q^2p^2q\cdot p^2\cdot qp^{-1}qp^{-1}\cdot qp^2qpq^{-1}p^{-1}q^2$ by (iii), (vii)

$= qpq^{-1}pq^2p^{-1}\cdot q^{-1}pq^{-1}p\cdot q^2pqp^2qp\cdot pq^2p^{-1}\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot pqpq^{-1}pqpq^{-1}p^2qp^{-1}qp^2q^2p^2$
$\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot qpq^{-1}p^{-1}q^2$ by (v), (iv)

$= qpq^{-1}pq^2p^{-1}q^{-1}pq^2p^2q^{-1}p^{-1}q^2pq^2\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot p^2q^{-1}p^{-1}\cdot qpqpq^{-1}p^2qp^{-1}qp^2q^2p^{-1}qpq^2$
$pq^{-1}p^{-1}q^2$ by (iii), (iv), (vii)

$= qpq^{-1}pq^2p^{-1}q^{-1}pq^2p^2q^{-1}p^{-1}q^{-1}\cdot q^{-1}pq^{-1}p\cdot q\cdot p^{-1}q^{-1}p^2q^{-1}p^{-1}q^2\cdot p^{-1}\cdot p^{-1}qp^{-1}qp^2q^2p^{-1}qpq^2$
$pq^{-1}p^{-1}q^2$ by (iv), (vii)

$= qpq^{-1}pq^2p^{-1}q^{-1}pq^2p^2\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot q^2\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot q^{-1}p^{-1}qpq^2pq^{-1}p^{-1}q^2$ by (iv), (vii)

$= qpq^{-1}pq^2p^{-1}q^{-1}pqpq^{-1}p\cdot q^{-1}p^{-1}qpq^{-1}p^{-1}qp\cdot q^2pq^{-1}p^{-1}q^2$ by (iii)

$= qpq^{-1}pq^2p^{-1}q^{-1}\cdot pqpq^2pqp\cdot p^2q^{-1}\cdot pq^{-1}pq^{-1}\cdot p^{-1}q^2$ by (vi)

$= qpq^{-1}pq^2\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot p^{-1}q^2pq^2$ by (vii), (iv)

$= 1$ by (iii), (iv).

(31)

$Dbac^{-1}b^{-1}a^{-1}b^{-1}D^{-1}c^{-1}b^{-1}Fc^{-1}b^{-1}cb^{-1}Fac^{-1}b^{-1}a^{-1}c^{-1}b$

$= q^2pq^2p^{-1}q^2\cdot pq^{-1}pq^{-1}\cdot p^2qpq^{-1}p^2qpq^{-1}p^2\cdot qp^{-1}qp^{-1}\cdot q^2pq^2p^{-1}q^2p^{-1}\cdot qp^2qpq^2p\cdot q^2pq^2p^{-1}$

$$qp^2q^{-1}pqpq^{-1}p^{-1}qp^{-1}qpq^2pq^2\cdot pq^{-1}pq^{-1}\cdot p^2qpq^{-1}p^{-1}qp^{-1}q^{-1}p^{-1}qpq^{-1}p^2$$

$$= q^2pq^2p^{-1}q^{-1}p^{-1}\cdot qpqp\cdot q^{-1}p^2qpq^{-1}p^{-1}q^{-1}\cdot pqpq\cdot qpq^2p^2q^{-1}\cdot q^{-1}pq^{-1}p\cdot pqpq\cdot qp^{-1}qp^2q^{-1}pqpq^{-1}$$

$$\cdot p^{-1}qp^{-1}q\cdot pq^2pq^{-1}p^{-1}\cdot qpqp\cdot q^{-1}p^{-1}qp^{-1}q^{-1}p^{-1}qpq^{-1}p^2 \quad \text{by (iv), (vii)}$$

$$= q^2p^2qp^2qp^{-1}q\cdot pq^{-1}p^{-1}q^2p^{-1}q^{-1}p^{-1}\cdot qpq^2p^2q^{-1}p^{-1}qp^2q^{-1}\cdot p^{-1}qp^{-1}q\cdot p^2q^{-1}pqpq^2pq^{-1}p^2q^2p$$

$$\cdot q^{-1}p^2q^{-1}p^{-1}q^2p^{-1}\cdot qp^{-1}q^{-1}p^{-1}qpq^{-1}p^2 \quad \text{by (iii), (iv), (vii)}$$

$$= q^2p^2qp^2\cdot qp^{-1}qp^{-1}\cdot qpq^2\cdot pqpq\cdot pq^2p^2q^{-1}p^{-1}qp^2q^2pq^{-1}p^{-1}q^{-1}\cdot pqpq\cdot qpq^{-1}\cdot p^2q^2p^2q^2$$

$$\cdot pqp^2q^2p^{-1}q^{-1}p^{-1}qpq^{-1}p^2 \quad \text{by (iii), (iv), (vii)}$$

$$= q^2p^2qp^{-1}q^{-1}p^2\cdot qp^{-1}qp^{-1}\cdot p^{-1}q^{-1}p^{-1}qp^2q^2\cdot pq^{-1}p^{-1}q^2p^{-1}q^{-1}p^{-1}$$

$$\cdot qpqp^2q^2p^{-1}qp^2q^2p^{-1}q^{-1}p^{-1}qpq^{-1}p^2 \quad \text{by (v), (iv), (iii), (vii)}$$

$$= q^2p^2q\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot p\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot qp^2q^2p^{-1}qpq^2\cdot pqpq$$

$$\cdot pqp^2q^2p^{-1}qp^2q^2p^{-1}q^{-1}p^{-1}qpq^{-1}p^2 \quad \text{by (iv), (vii), (iii)}$$

$$= q^2p^2q^2p\cdot qp^{-1}qp^{-1}\cdot q^2p^2q^2\cdot pqpq\cdot pq^2p^{-1}qp^2q^{-1}pqpq^2pq^{-1}p^2 \quad \text{by (iii)}$$

$$= q^2p^2q^2p^2q^{-1}pqp^2\cdot qp^{-1}qp^{-1}\cdot qp^2q^{-1}pqpq^2pq^{-1}p^2 \quad \text{by (iii), (iv)}$$

$$= q^2p^2q^2p^2q^{-1}pq\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot pqpq^2pq^{-1}p^2 \quad \text{by (iv)}$$

$$= q^2p^2q^2p^2\cdot q^{-1}pq^2pqp^2q\cdot pq^2pq^{-1}p^2 = 1 \quad \text{by (vii), (v).}$$

(32)

$$D^{-1}aD^{-1}ab^{-1}aFac^{-1}abFac^{-1}$$

$$= pq^2p^{-1}q^2pq^2p^{-1}q^{-1}pqp^{-1}q^2pq^2p^{-1}q^{-1}pq^{-1}pqp^{-1}\cdot q^{-1}p^2\cdot q^{-1}p^{-1}(pq)^4\cdot qpq^{-1}p^{-1}\cdot (pq^{-1})^4$$

$$\cdot qp\cdot qp^{-1}q^{-1}p^2qp^2q^{-1}p^{-1}q^2pq^2pq^2\cdot pq^{-1}pq^{-1}\cdot p^2qp^{-1}q^{-1}p^2$$

$$= pq^2p^{-1}q^2pq^2p^{-1}q^{-1}pqp^{-1}q^2pq^2p^{-1}q^{-1}pq\cdot q^2pqp^2qp\cdot p^{-1}\cdot (q^{-1}p)^4p^{-1}q^2p^{-1}\cdot q^{-1}pq^{-1}p\cdot (p^{-1}q^{-1})^4$$

$$\cdot qp\cdot qp^{-1}qp^{-1}\cdot qpqp(p^{-1}q^{-1})^4\cdot q^{-1}pq^2pq^{-1}p^{-1}qpqp^{-1}q^{-1}p^2 \quad \text{by (iii), (iv), (vi)}$$

$$= pq^2p^{-1}q^2pq^2p^{-1}q^{-1}pqp^{-1}q^2pq\cdot qp^{-1}q^{-1}pqp^{-1}q^{-1}pqp^{-1}\cdot pq^{-1}pq^{-1}p^{-1}\cdot q^2p^2q^2p^2\cdot q^2p^2q$$

$$\cdot q^{-1}p^2q^{-1}p^{-1}q^2p^{-1}\cdot p^{-1}q^{-1}p^2qpq^{-1}pq^2pq^{-1}p^{-1}qpqp^{-1}q^{-1}p^2 \quad \text{by (iii), (iv), (vii)}$$

$$= pq^2p^{-1}q^2pq^2p^{-1}q^{-1}pqp^{-1}\cdot q^2p^2(p^{-1}q)^4q^2p^2\cdot q\cdot pq^{-1}pq^{-1}\cdot q^{-1}pq^{-1}p^{-1}qpqp^{-1}q^{-1}p^2$$

$$\text{by (iii), (iv), (vii)}$$

$$= pq^2p^{-1}q^2pq^2\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot qp\cdot pqpq^2pqp\cdot p\cdot pq^{-1}pq^{-1}\cdot p^{-1}qpqp^{-1}q^{-1}p^2 \quad \text{by (iii), (v), (iv)}$$

$$= pq^2p^{-1}q^{-1}p^{-1}(pq^{-1})^4\cdot qpqp^2qpq\cdot p^{-1}q^{-1}\cdot p^2 \quad \text{by (iv), (vii), (iii)}$$

$$= 1 \quad \text{by (iv), (vii), (iii).}$$

(33)

$$Fac^{-1}b^{-1}a^{-1}cb^{-1} = pq^2pq^2pq^2\cdot pq^{-1}pq^{-1}\cdot pq^{-1}\cdot qpqp\cdot q^{-1}p^2qpq^{-1}p^{-1}qp^{-1}q^{-1}p^{-1}$$
$$= pq^2pq^2p\cdot q^{-1}p^2q^{-1}p^{-1}q^2p^{-1}\cdot p^{-1}qpq^{-1}p^{-1}qp^{-1}q^{-1}p^{-1} \quad \text{by (iii), (iv)}$$
$$= pq^2\cdot pq^2p^2q^2p\cdot qp^2qp^{-1}qpq^{-1}p^{-1}\cdot qp^{-1}q^{-1}p^{-1} \quad \text{by (vii)}$$
$$= pq^2p^{-1}q^2p^2q\cdot qp^{-1}qp^{-1}\cdot p^{-1}qp^{-1}q\cdot qp^2qpq^2p\cdot q^{-1}pq^2p^{-1} \quad \text{by (v), (iv), (vii)}$$
$$= pq^2p^{-1}q^2p^2qpq^2\cdot (qpq^{-1}p^{-1})^2\cdot q^2p^{-1}q^{-1}p^2q^2pq^2p^{-1} \quad \text{by (iv), (vii).}$$

Then Lemma 2 (33) follows by (vi).

(34)

$$D^{-1}Fac^{-1}abcb^{-1} = pq^2p^{-1}q^2pq^2p^2q^2p^{-1}q^2p^{-1}q^2p^{-1}qpq^{-1}p^2q\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot qp^{-1}qp^{-1}\cdot q^2 \cdot qp^{-1}qp^{-1}q^{-1}p^{-1}$$
$$= pq^2p^{-1}q^2pq^2p^2q^2p^{-1}q^2p^{-1}q^2p^{-1}q\cdot pq^{-1}pq^{-1}\cdot q\cdot pq^2pqp^2q\cdot q\cdot p^{-1}q^{-1}p^{-1}q^2pq^2p^{-1} \quad \text{by (iii), (iv)}$$
$$= pq^2p^{-1}q^2pq^2p^2q^2p^{-1}q^2p^{-1}q^2\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot qpq^{-1}\cdot p^{-1}qpq^{-1}p^{-1}qpq^{-1} \cdot pq^2p^{-1}q^2pq^2p^{-1} \quad \text{by (iv), (vii)}$$
$$= pq^2p^{-1}q^2pq^2p^2q^2p^{-1}qp^{-1}\cdot pqp^{-1}q^{-1}pqp^{-1}q^{-1}(qp^{-1})^4\cdot pq^{-1}p \cdot q^2p^2q^2p^{-1}q^2pq^2p^{-1} \quad \text{by (iii), (vi).}$$

Obviously we have Lemma 2 (34) by (iv), (vi).

(35)

$$caFab^{-1}a = p^2qpq^{-1}\cdot pqpq\cdot pq^2pq^2\cdot pq^{-1}pq^{-1}\cdot pqp^{-1}q^{-1}p^{-1}\cdot qpq^{-1}p^{-1}$$
$$= p\cdot pqpq\cdot qp^{-1}qp^{-1}\cdot qp^{-1}\cdot p^{-1}\cdot p^{-1}q(qp)^4p^{-1}q\cdot qp^{-1} \quad \text{by (iii), (iv), (vii)}$$
$$= pq^{-1}p^{-1}q\cdot qp^{-1}q^{-1}p\cdot q^{-1}pqp^{-1} \quad \text{by (iii), (iv).}$$

Then Lemma 2 (35) follows by (vi).

(36)

$$dad^{-1}Dbacb^{-1}Fac^{-1}b^{-1}a^{-1}b^{-1}D^{-1}$$
$$= q^2pq^2p^{-1}qp\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot (pq^{-1})^4\cdot qpqpq^{-1}\cdot pqpq^2pqpp^{-1}q^{-1}\cdot (p^{-1}q)^4q^{-1}p^2q^2pq^2\cdot pq^{-1}pq^{-1} \cdot pq^{-1}\cdot qpqp\cdot q^{-1}p^2\cdot qp^{-1}qp^{-1}\cdot q^2pq^2p^{-1}q^2$$
$$= q^2pq^2p^{-1}q\cdot p^2qpq^2pq\cdot p\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot q^2p^{-1}q^{-1}\cdot p^2q^2p^2q^2\cdot pq^{-1}\cdot p^2q^{-1}p^{-1}q^2p^{-1}q^{-1} \cdot pqpq^2p^{-1}q^2 \quad \text{by (iv), (vii), (iii)}$$
$$= q^2pq^2p^{-1}\cdot (p^{-1}q)^4pq^2p^{-1}q^2 = 1 \quad \text{by (iv), (iii), (vii).}$$

(37)

$$eae^{-1}a^{-1}b^{-1}D^{-1}cb^{-1}Fac^{-1}b^{-1}a^{-1}b^{-1}$$

$$= pq^2qp^2p^2q\cdot pq^{-1}pq^{-1}\cdot q^{-1}\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot p^2\cdot qp^{-1}qp^{-1}\cdot q^2pq^2p^{-1}q^2p^2\cdot qpq^{-1}\cdot p^{-1}qp^{-1}q\cdot pq^2pq^2$$
$$\cdot pq^{-1}pq^{-1}\cdot p^2qpq^{-1}p^2qp^{-1}q^{-1}p^{-1}$$

$$= pq^2\cdot pq^2p^2q^2p^{-1}\cdot qpqp\cdot q^2p^{-1}q\cdot qp^2qpq^2p\cdot q^{-1}p^2q^2pq^{-1}p^{-1}\cdot qpqp\cdot q^{-1}p^2qp^{-1}q^{-1}p^{-1}$$
$$\text{by (iv), (iii)}$$

$$= pq^2p^{-1}q^2p^2\cdot qp^{-1}qp^{-1}\cdot qp^{-1}q^2p^{-1}q^{-1}\cdot p^2q^2p^2q^2\cdot p\cdot q^{-1}p^2q^{-1}p^{-1}q^2p^{-1}\cdot p^{-1}qp^{-1}q^{-1}p^{-1}$$
$$\text{by (iv), (iii), (vii), (v)}$$

$$= pq^2p^{-1}q^2\cdot p^{-1}qp^{-1}q\cdot p^{-1}qp^2\cdot qp^{-1}qp^{-1}\cdot q^{-1}p^{-1} = 1 \quad \text{by (v), (iv), (vii).}$$

(38)

$$de\cdot a\cdot e^{-1}d^{-1}Dbab = q^2p^2q^2p^2\cdot q\cdot pq^{-1}pq^{-1}\cdot p\cdot p^{-1}q^{-1}p^{-1}q^2p^{-1}q^{-1}p^{-1}\cdot pq^{-1}pq^{-1}pq^{-1}\cdot p^2qp^2q^{-1}p^2$$
$$= p^2q^2\cdot pqpq\cdot pq^2\cdot pqpq\cdot pqp^2q^{-1}p^2 = 1 \quad \text{by (iii), (iv), (vii), (v).}$$

(39)

$$dbd^{-1}Dbabca^{-1}Fbc^{-1} = q^2\cdot pq^{-1}pq^{-1}\cdot p^2q^2\cdot pq^{-1}pq^{-1}\cdot p^2q\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot (qp^{-1})^4\cdot pqp^{-1}q^2p^2$$
$$\cdot (pq^{-1})^4q^{-1}\cdot q^{-1}p^{-1}q^2p^{-1}q^{-1}p^2$$
$$= q^{-1}p^{-1}qpq^{-1}p^{-1}\cdot qpq^2pqp^2\cdot qp^{-1}\cdot q^2p^2q^2p^2\cdot qpq^2pq$$
$$= q^{-1}p^{-1}q\cdot (pq^{-1})^4\cdot q^{-1}pq = 1 \quad \text{by (iii), (iv), (v), (vii).}$$

(40)

$$ebe^{-1}a^{-1}ba^{-1}F = pq^2pq^2p^{-1}\cdot qpq^2pq\cdot (q^{-1}p^{-1})^4p^2qpq^2p(p^{-1}q)^4\cdot qp^2\cdot pq^2p^{-1}q^2p^{-1}$$
$$= 1 \quad \text{by (iii), (iv), (vii).}$$

(41)

$$de\cdot b\cdot e^{-1}d^{-1}Dbabca^{-1}Fbc^{-1}bcFbc$$

$$= q^2p^2q^2p^{-1}qpqp^{-1}q^2p^{-1}q^2\cdot pq^{-1}pq^{-1}\cdot pq^{-1}\cdot qpq\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot (qp^{-1})^4\cdot pqp^{-1}q^2p^2q^2p$$
$$\cdot p^{-1}q^{-1}(q^{-1}p)^4\cdot p^{-1}q^2p^{-1}q^{-1}p^{-1}\cdot qp^2q^{-1}pq^2p^{-1}q^2\cdot p^{-1}q^2p^{-1}q^{-1}p^2q^{-1}\cdot p$$

$$= q^2p^2q^2p^{-1}qpqp^{-1}q^2p^{-1}q^{-1}pq^{-1}p^{-1}\cdot q^2p^2q^2p^2\cdot q^2p^2qpq^2\cdot pqpq\cdot p^2q^{-1}pq^2p^{-1}q^{-1}p^2qpq^2p^2$$
$$\text{by (iii), (iv), (vii)}$$

$$= q^2p^2q^2p^{-1}qpqp^{-1}q^2p^{-1}\cdot q^{-1}pq^{-1}p\cdot q^{-1}pqp^{-1}\cdot q^{-1}pq^{-1}p\cdot q^2p^{-1}q^{-1}\cdot p^2qpq^2p^2 \quad \text{by (v), (iii)}$$

$$= q^2p^2q^2p^{-1}qpqp^{-1}\cdot q^2p^2q^2p^2\cdot q\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot p^2qpq^2p^2$$

$$= q^2p^2q^2p^{-1}\cdot qpqp\cdot q^2p^{-1}qp^{-1}q\cdot pq^2p^2 = 1 \quad \text{by (iii), (iv), (v).}$$

(42)

$$dcd^{-1}Dbabca^{-1}FD^{-1} = q^2pq^2pq\,pq^{-1}pq^{-1}\cdot q^{-1}\cdot pq^{-1}pq^{-1}\cdot p^2q\cdot p^{-1}q^{-1}p^{-1}q^{-1}(qp^{-1})^4\cdot pq$$
$$\cdot p^{-1}q^2p^2q^2p\cdot q^2p^{-1}q^2$$
$$= q^2pq^2pq^2p^{-1}q\cdot(p^2qpq^2pq)^2\cdot q^{-1}pq^2p^{-1}q^2p^{-1}q^2 = 1$$

by (iii), (iv), (vi).

(43)

$$ece^{-1}babca^{-1}Fbc^{-1}Dbc^{-1}$$
$$= qp^2\cdot pq^{-1}pq^{-1}\cdot p^{-1}q^2p^{-1}\cdot q^{-1}pq^{-1}p\cdot pq\cdot p^{-1}q^{-1}p^{-1}q^{-1}(qp^{-1})^4\cdot pqp^{-1}q^2p^2q(q^{-1}p)^4$$
$$\cdot p^{-1}q^2p^{-1}q^{-1}p^2q^{-1}\cdot q^{-1}pq^{-1}\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot(pq^{-1})^4\cdot q^2p^{-1}q^{-1}p^2$$
$$= pq^{-1}p^{-1}q^{-1}p^2q^2p^2q^{-1}p^{-1}\cdot qpq^2pqp^2\cdot qp^{-1}\cdot q^2p^2q^2p^2\cdot qpq^2\cdot pq^{-1}pq^{-1}\cdot pqpq^{-1}p^{-1}q^{-1}p^2$$

by (v), (iii), (iv), (vii)

$$= pq^{-1}p^{-1}q^{-1}p^2q^2p\cdot(pq^{-1})^4\cdot p^{-1}q^2pq^{-1}p^{-1}q^{-1}p^2 = 1$$ by (vii), (iii), (iv).

(44)

$$dece^{-1}d^{-1}\cdot b^{-1}cb^{-1}D^{-1} = q^2p\cdot qp^{-1}qp^{-1}\cdot p^{-1}\cdot q^2p^2q^2p^2\cdot qp^{-1}q^{-1}pqpq^{-1}p^{-1}\cdot qp^{-1}qp^{-1}\cdot q^2pq^2p^{-1}q^2$$
$$= q^2pqp^{-1}q^{-1}p^2q^{-1}p^{-1}q^{-1}\cdot pqpq^2pqp\cdot q^2p^{-1}q^2$$
$$= q^2pqp^{-1}q^{-1}p^2\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot q^{-1}p^{-1}q^{-1}\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot q^2p^{-1}q^2 = 1$$

by (iii), (iv), (vii).

(47)

$$e^2cFbabca^{-1}Fbc^{-1}Dbc^{-1}$$
$$= pq^2pq^2p^{-1}q^{-1}p^{-1}\cdot(pq^{-1})^4\cdot q^{-1}p^{-1}q^2p^{-1}q^2p^{-1}q^{-1}pq^{-1}p^2q\cdot p^{-1}q^{-1}p^{-1}q^{-1}\cdot(pq^{-1})^4pqp^{-1}q^2p^{-1}$$
$$\cdot(q^{-1}p)^4\cdot p^{-1}qp^{-1}q^2p^{-1}q^{-1}p^2q^2pq^2p^{-1}\cdot q^{-1}\cdot(q^{-1}p)^4p^{-1}\cdot q^2p^{-1}q^{-1}p^2$$
$$= pq^2\cdot pq^{-1}pq^{-1}\cdot p^{-1}q^2p^{-1}\cdot q^{-1}pq^{-1}p\cdot pq^2pqp^2q\cdot p^{-1}\cdot q^2p^2q^2p^2\cdot qpq^2pq^{-1}pqp^{-1}q^{-1}p^{-1}qpq$$

by (iii), (iv), (vii)

$$= pq^{-1}p^{-1}q^{-1}p^2q^2p^2\cdot q^{-1}pq^{-1}p\cdot q^{-1}pq^2pq^{-1}pqp^{-1}q^{-1}p^{-1}qpq$$
$$= pq^{-1}p^{-1}q^{-1}p^2q^2\cdot pq^{-1}pq^{-1}\cdot pqp^{-1}q^{-1}p^{-1}qpq$$
$$= pq^{-1}\cdot p^{-1}q^{-1}p^2q^{-1}p^{-1}q^2\cdot p^{-1}q^{-1}p^{-1}\cdot qpq = 1$$ by (iii), (iv), (vii).

(46)

$$dede^{-1} = (de)^2\cdot e^2\ .$$

But by (v) we have $dede = (q^2p^2)^4 = 1$. Hence Lemma 2 (46) follows by Lemma 2 (47).

(48)

$$de^{-2}d^{-1}\cdot debe^{-1}d^{-1}\cdot dece^{-1}d^{-1}\cdot dbd^{-1}\cdot dcd^{-1} = de^{-1}bce^{-1}bcd^{-1}$$
$$= q^2pq^2p\left(q^2p^{-1}q^{-1}p^2q^{-1}p^{-1}\right)^2\cdot p^{-1}q^2p^{-1}q^2 = 1$$

Hence Lemma 2 (48) follows by Lemma 2 (41), (44), (39), (42).

(45)

$$d^2\cdot ede^{-1}d^{-1}\cdot de^2d^{-1} = d\cdot(de)^2\cdot d^{-1} = 1 .$$

Lemma 2 (48) yields Lemma 2 (45).

(49)

$$\left(dbc^{-1}\right)^2 = \left(q^2\cdot pq^{-1}pq^{-1}\cdot pq\cdot p^{-1}q^{-1}p^2\right)^2 = \left(q^{-1}p^{-1}q^2p^{-1}q^{-1}p^2\right)^2 = 1 \text{ by (iv), (vii),}$$
$$(db)^4 = \left(q^2\cdot pq^{-1}pq^{-1}\cdot p^2\right)^4 = \left(q^{-1}p^{-1}qp\right)^4 = 1 \text{ by (iv), (vi),}$$
$$(dba)^2 = \left(q^2\cdot pq^{-1}pq^{-1}pq^{-1}\cdot qpqp\cdot q^{-1}p^{-1}\right)^2$$
$$= \left(q^{-1}p^2q^{-1}p^{-1}q^2p^{-1}\right)^2 = 1 \text{ by (iii), (iv), (vii).}$$

LEMMA 3. *D operates on the generators of H_g as follows:*

(1) $D^{-1}a^2D = bab^2a^{-1}c^{-1}bca^{-1}b^2a^{-1}c^{-1}b^{-1}cb^{-1}$;

(2) $D^{-1}b^2D = bc^{-1}bcab^2ac^2b^2c^2a^{-1}b^2a^{-1}c^{-1}b^{-1}cb^{-1}$;

(3) $D^{-1}c^2D = bc^{-1}bcab^2ac^2a^{-1}b^2a^{-1}c^{-1}b^{-1}cb^{-1}$;

(4) $D^{-1}abD = bac^2a^{-1}b^2a^{-1}c^{-1}b^{-1}cb^{-1}$;

(5) $D^{-1}baD = bc^{-1}bcab^2ac^2a^{-1}c^{-1}bca^{-1}b^2a^{-1}c^{-1}b^{-1}cb^{-1}$;

(6) $D^{-1}acD = bab^2a^{-1}c^{-1}bc^{-1}ab^2cab^2ac^2a^{-1}b^2a^{-1}c^{-1}b^{-1}cb^{-1}$;

(7) $D^{-1}caD = bc^{-1}b^{-1}a^{-1}c^2a^{-1}b^2a^{-1}c^{-1}b^{-1}cb^{-1}$;

(8) $D^{-1}bcD = bc^{-1}b^{-1}a^{-1}b^{-1}a^{-1}c^{-1}b^{-1}cb^{-1}$;

(9) $D^{-1}cbD = bc^{-1}bcab^2ac^2a^{-1}b^{-1}ab^2cab^2ac^2b^2c^2a^{-1}b^2a^{-1}c^{-1}b^{-1}cb^{-1}$;

(10) $D^2 = bc^{-1}bcab^2ac^2a^{-1}c^{-1}bc^{-1}b^{-1}a^{-1}$.

Proof. (8) follows by Lemma 2 (3), (4).

(4) By Lemma 2 (6) we have $D^{-1}abc^{-1}b^{-1}bc^{-1}b^{-1}a^{-1}D = bac^2a^{-1}b^{-1}$. So Lemma 2 (5) and Lemma 3 (8) yield Lemma 3 (4).

(3) follows by Lemma 2 (6) and Lemma 3 (4).

(7) By Lemma 2 (5), (7), (8) we have $D^{-1}cab^{-1}a^{-1}D = bc^{-1}b^{-1}a^2b^{-1}$. Lemma 3 (4) yields Lemma 3 (7).

(2) By Lemma 2 (8) we have $D^{-1}ab^2a^{-1}D = bab^2a^{-1}b^{-1}$. Lemma 3 (4) yields Lemma 3 (2).

(10) By Lemma 2 (1), (7) we have $D^{-1}abDD^2ab = bc^{-1}bc^{-1}$. Lemma 3 (4) yields Lemma 3 (10).

(5) By Lemma 2 (8) we have $D^2D^{-1}baDD^{-1}b^{-1}a^{-1}D = aba^{-1}b^{-1}$. Lemma 3 (4), (10) yield Lemma 3 (5).

(1) By Lemma 2 (8), (9) we have $D^{-1}ab^2a^{-1}D = bab^2a^{-1}b^{-1}$. Lemma 3 (4), (5) yield Lemma 3 (1).

(9) By Lemma 2 (5) we have $D^{-1}abDD^{-1}cbDD^{-1}b^2D = abcb^{-1}$. Lemma 3 (4), (2) yield Lemma 3 (9).

(6) We have the identity $D^{-1}acD = D^{-1}a^2DD^{-1}a^{-1}c^{-1}DD^{-1}c^2D$. Lemma 3 (1), (3), (7) yield Lemma 3 (6).

LEMMA 4. *In* $H = \langle a, b, c\rangle$ *we have*

(1) $b^2cab^{-2} = a^2ca^{-1}$;

(2) $b(c^{-1}a)^2b^{-1} = a^{-1}(c^{-1}a)^2a$;

(3) $b(ac^{-1})^2b^{-1} = (c^{-1}a)^2$;

(4) $b(ac)^2b^{-1} = (ca)^2$;

(5) $b(ca)^2b^{-1} = a^{-1}(ca)^2a$;

(6) $(bca^2c^{-1}a^2b^{-1}a^2)^2 = 1$;

(7) $b^{-1}a(ac)^2a^{-1}(ac^{-1})^2b = (a^{-1}c)^2(ac)^2$;

(8) $ba(ac)^2a^{-1}(ac^{-1})^2b^{-1} = (c^{-1}a)^2(ac)^2$;

(9) $(ca)^2 \rightleftarrows (a^{-1}c)^2$, $(ca)^2 \rightleftarrows (ac^{-1})^2$, $(ac)^2 \rightleftarrows (a^{-1}c)^2$, $(ac)^2 \rightleftarrows (ac^{-1})^2$;

(10) $b^{-1}(ac^{-1})^2b = a(ac^{-1})^2a^{-1}$;

(11) $c^{-1}a \rightleftarrows (ac^{-1})^4$;

(12) $b^{-1}(a^{-1}c)^2c^{-1}(ac)^2cb = c(ca)^2c^{-1}(ca^{-1})^2$;

(13) $b^{-1}c(ac^{-1})^2c^{-1}b = ac(ac^{-1})^2c^{-1}a^{-1}$;

(14) $(aca^{-1}cac^{-1})^4 = 1$;

(15) $aca^{-1}c^{-1}ac \rightleftarrows (ca)^2$;

(16) $(ac)^2 \rightleftarrows (ca)^2$;

(17) $bc^{-1}(ac)^2c(c^{-1}a)^2b^{-1} = a^{-1}c^{-1}(ac)^2c(c^{-1}a)^2a$;

(18) $bc^{-1}(ac)^2cb^{-1} = a^{-1}c^{-1}(ac)^2ca$;

(19) $ba^{-1}(a^{-1}c)^2ab^{-1} = a(ca^{-1})^2a^{-1}$;

(20) $ba^{-1}c^{-1}(ac)^2cab^{-1} = c^{-1}a^{-1}(ca)^2ac$;

(21) $b^{-1}c(ca)^2c^{-1}b = c^{-1}a^{-1}(ca)^2ac$.

Proof. (1) By Lemma 2 (10) we have

$$p^2Dbacb^{-1}D^{-1}aba^{-1}c^{-1}b^{-1}a^{-1}p^{-2} = 1 .$$

Lemma 1 (1)-(5) and Lemma 2 (5) yield $D^{-1}caD = bc^{-1}b^{-1}a^{-1}\cdot ca\cdot abcb^{-1}$. A comparison with Lemma 3 (7) yields Lemma 4 (1).

(2) and (3). By Lemma 4 (1) we have

$$p^2b^2cab^2p^{-2} = p^2a^2ca^{-1}p^{-2} \Rightarrow b^{-1}a^{-1}c^{-1}b^2a^{-1}c^{-1}b^{-1} = c^{-1}ac^{-1}a$$

by Lemma (1)-(5). Use Lemma 4 (1) in both possible ways obtaining Lemma 4 (2) and Lemma 4 (3).

(4) By Lemma 3 (1) we have

$$D^{-2}a^2D^2 = D^{-1}bab^2a^{-1}c^{-1}bca^{-1}bca^{-1}b^2a^{-1}c^{-1}b^{-1}cb^{-1}D .$$

By Lemma 3 (5), (2), (7), (8), (4), (9) follows

$$a^{-1}c^{-1}bc^{-1}ab^2cab^2ac^2a^{-1}b^{-1}ab^2cab^2ac^2 = 1 .$$

Lemma 4 (1) and Lemma 2 (2) yield Lemma 4 (4).

(5) By Lemma 4 (4) we have

$$p^2b(ac)^2b^{-1}p^{-2} = p^2(ca)^2p^{-2} .$$

Lemma 1 (1)-(5) yield Lemma 4 (5).

(6) follows by Lemma 3 (1), Lemma 4 (2) and $a^4 = 1$.

(7) By Lemma 4 (2), (4) we have

$$b(a^{-1}c)^2b^{-1}b(ac)^2b^{-1} = a^{-1}\cdot a^{-1}ca^{-1}c\cdot acac\cdot a \ .$$

Lemma 2 (2) yields Lemma 4 (7).

(8) By Lemma 4 (1) we have

$$D^{-1}b^2cab^2ac^{-1}a^2D = 1 \ .$$

Using Lemma 3 (2), (7), (6), (3), (1) we have

$$c^{-1}b^2c^2a^{-1}b^2a^{-1}c^{-1}b^2a^{-1}b^2c^2a^{-1}b^2a^{-1}c^{-1}b^{-1}cab^2a^{-1}c^{-1}bc^{-1}abcab^2a^{-1}c^{-1}b = 1 \ .$$

By Lemma 4 (1), (7) follows

$$ca^{-1}ca^2bca^2c^{-1}a^2b^{-1}a^{-1}caca^{-1} = 1 \ .$$

Lemma 4 (6) and Lemma 2 (2) yield Lemma 4 (8).

(9) By Lemma 4 (8), (3), (4), (1) we have

$$a^2ca^2c^{-1} = b^{-1}(c^{-1}a)^2bb^2b(ac)^2b^{-1}b^2 = (ac^{-1})^2a^2caca^{-1} \ .$$

Hence we have

$$(ac)^2 \rightleftarrows (c^{-1}a)^2 \ \Rightarrow \ p^2(ac)^2p^{-2} \rightleftarrows p^2(c^{-1}a)^2p^{-2} \ \Rightarrow \ (ca)^2 \rightleftarrows (a^{-1}c)^2$$

by Lemma 1 (1)-(5).

Lemma 4 (7) and Lemma 2 (2) yield

$$b^{-1}a^2ca^2c^{-1}bb^{-1}a^2ca^2c^{-1}b = (a^{-1}c)^4 \ .$$

By Lemma 4 (2) follows

$$(ca)^2 \rightleftarrows (ac^{-1})^2 \ .$$

Applying p^2 , we obtain the missing relation.

(10) By Lemma 4 (2) we have

$$b^2c^{-1}ac^{-1}ab^2 = ba^{-1}c^{-1}a^{-1}c^{-1}b^{-1}bca^2c^{-1}a^2b^{-1} \ .$$

Lemma 4 (5), (8), (9) and Lemma 2 (2) yield

$$b^2c^{-1}ac^{-1}ab^2 = a^{-1}cacaca^{-1}cacac \ .$$

Applying Lemma 2 (2) once again, we obtain

$$b^2(c^{-1}a)^2b = a(ac^{-1})a^{-1} \ .$$

By Lemma 4 (3) we obtain Lemma 4 (10).

(11) By $a^4 = 1$ we have

$$(a^{-1}a^{-1}c^{-1}a^{-1}a^{-1}ca^{-1}a^{-1})^2 = 1 \ .$$

Lemma 4 (10), Lemma 2 (2) yield

$$(ca^{-1}ca^{-1}ca^{-1}ca)^2 = 1 .$$

Applying p^2 we obtain Lemma 4 (11).

(12) By Lemma 4 (8) we have

$$p^2ba^2ca^2c^{-1}b^{-1}p^{-2} = p^2(c^{-1}a)^2(ac)^2p^{-2} .$$

Lemma 1 (1)-(5), Lemma 4 (9) yield Lemma 4 (12).

(13) By Lemma 4 (4) and Lemma 4 (12) follows

$$b^{-1}cacaa^{-1}c^2ac^2b = acacc^2ac^2a^{-1} .$$

(14) By Lemma 2 (2), Lemma 3 (6), Lemma 4 (1) we have

$$(a^2c^{-1}a^2b^{-1}c^{-1}a^{-1}c^2a^2bc)^4 = 1 .$$

Lemma 4 (8) yields Lemma 4 (14).

(15) By Lemma 4 (4) and Lemma 2 (2) we have

$$D^{-1}acacb^{-1}cacabD = 1 .$$

Use Lemma 3 (6), (2), (8), (4) and Lemma 4 (1) to conclude

$$bca^2c^{-1}a^2b^{-1}c^{-1}a^{-1}c^2a^2b^{-1}a^2ca^2c^{-1}bc^{-1}a^{-1}cb^2ca$$
$$\cdot b^{-1}a^{-1}c^{-1}b^2\cdot b^{-1}a^2ca^2c^{-1}bc^{-1}a^{-1}c^2a^2bcaca^2 =$$

By Lemma 4 (8), (7), (1) we have

$$b^{-1}(ac^{-1})^2c(ca^{-1})^2c^{-1}ca^2c^{-1}a^2bcaca^{-1}caca^{-1}\cdot ca^2c^2aca^{-1}cac^2a^2ca^{-1} = 1 .$$

By Lemma 4 (10), (7), (13) follows

$$a^2c^{-1}aca^{-1}cac^{-1}acaca^2acaca^{-1}ca^{-1}cc^{-1}a^{-1}c^2ac\cdot a^{-1}cac^2a^2ca^{-1} = 1 .$$

By Lemma 2 (2) and Lemma 4 (9) we have

$$a^2c^{-1}aca^{-1}cac^2a^2cac^{-1}aca^{-1}cac^2a^2ca^{-1} = 1 .$$

Apply the automorphism induced by p^2 obtaining

$$(c^{-1}aca^{-1}cac^{-1}aca^{-1}caa^{-1}c^{-1}a^{-1}c^{-1})^2 = 1 .$$

By Lemma 4 (14) and Lemma 2 (2) we have

$$(c^{-1}aca^{-1}ca)^2 \neq (ca)^2 .$$

Then Lemma 4 (9) implies Lemma 4 (15).

(16) By Lemma 4 (15) and Lemma 2 (2) we have

$$D^{-1}aca^{-1}c^{-1}accacac^{-1}a^{-1}cac^{-1}a^{-1}cacaD = 1 .$$

By Lemma 3 (6), (7) and Lemma 4 (1) we have

$$a^2b^{-1}a^2ca^2c^{-1}bc^{-1}a^{-1}cab^{-1}a^2ca^2c^{-1}bc^{-1}a^{-1}c^2acac^{-1}ac$$
$$ba^2ca^2c^{-1}b^{-1}a^{-1}c^{-1}acba^2ca^2c^{-1}b^{-1}a^{-1}cac = 1 .$$

Lemma 4 (7), (8) imply

$$cacaca^{-1}cac^{-1}a^{-1}cac^{-1}acac^{-1}a^2c^{-1}a^2caca^{-1}c^{-1}ac\left(c^{-1}a\right)^2(ac)^2a^{-1} = 1 .$$

Lemma 4 (9) and Lemma 2 (2) yield

$$a^{-1}c^{-1}a^{-1}cac^{-1}acac^{-1}ac\left(c^{-1}a\right)^2(ac)^2a^{-1}c^2a^{-1}c^{-1}ac^{-1}\cdot(ca)^2\left(ca^{-1}\right)^2a^{-1} = 1 .$$

Lemma 4 (9) and Lemma 2 (2) imply

$$(ca)^2c^{-1}a^{-1}\left(ac^{-1}\right)^2(ac)^2\left(ca^{-1}\right)^2ac(ca)^2(ac)^2 = 1 .$$

Hence Lemma 4 (9) and Lemma 2 (2) imply Lemma 4 (16).

(17) By Lemma 4 (4), (10) we have

$$p^2b^{-1}ac^{-1}ac^{-1}acacbp^{-2} = p^2a^2c^{-1}a^2cp^{-2} .$$

By Lemma 1 (1)-(5) follows Lemma 4 (17).

(18) By Lemma 4 (17) we have

$$bc^2a^{-1}c^{-1}a^{-1}cc^{-1}ac^{-1}ab^{-1} = a^{-1}c^2a^{-1}c^2a^2 .$$

Lemma 4 (2) and Lemma 2 (2) yield Lemma 4 (18).

(19) By Lemma 4 (8), (6) we have

$$ba^2ca^2c^{-1}cacab^{-1} = c^{-1}ac^{-1}aacaca^{-1}(ca)^2a .$$

Lemma 4 (9) implies Lemma 4 (19).

(20) By Lemma 2 (2) we have

$$bcabcabcabca = 1 .$$

Lemma 4 (1) yields

$$b^{-1}a^2ca^{-1}ba^{-1}\left(a^{-1}c\right)^2aa^{-1}c^{-1}b^{-1}cabca = 1 .$$

Lemma 4 (19) implies

$$ba^{-1}c^{-1}b^{-1}cabcab^{-1} = a^2c^{-1}ac^2a^2 .$$

Raise this relation to the second power obtaining

$$ba^{-1}c^{-1}b^{-1}(ca)^2bcab^{-1} = a^2c^{-1}\cdot acac\cdot ca^2 .$$

Use Lemma 4 (4), (9), (16) and Lemma 2 (2) to conclude Lemma 4 (20).

(21) By Lemma 4 (20) we have

$$b^{-1}b^2a^{-1}c^{-1}(ac)^2cab^2b = c^{-1}a^{-1}(ca)^2ac .$$

Lemma 4 (1), (5), (4), (9) yield Lemma 4 (21).

LEMMA 5. *The following rules of conjugation apply:*

(1) $a(ac)^2a^{-1} = a(ac)^2a^{-1}$;

(2) $b(ac)^2b^{-1} = (ca)^2$ *by Lemma* 4 (4);

(3) $c(ac)^2c^{-1} = (ca)^2$;

(4) $a^2(ac)^2a^2 = a^{-1}(ca)^2a$;

(5) $b^2(ac)^2b^2 = a^{-1}(ca)^2a$ *by Lemma* 4 (4), (5);

(6) $c^2(ac)^2c^2 = c(ca)^2c^{-1}$;

(7) $a^{-1}(ac)^2a = (ca)^2$;

(8) $b^{-1}(ac)^2b = a(ac)^2a^{-1}$ *by Lemma* 4 (4), (1) ;

(9) $c^{-1}(ac)^2c = c^{-1}(ac)^2c$;

(10) $a(ca)^2a^{-1} = (ac)^2$;

(11) $b(ca)^2b^{-1} = a^{-1}(ca)^2a$ *by Lemma* 4 (5);

(12) $c(ca)^2c^{-1} = c(ca)^2c^{-1}$;

(13) $a^2(ca)^2a^2 = a(ac)^2a^{-1}$;

(14) $b^2(ca)^2b^2 = a(ac)^2a^{-1}$ *by Lemma* 4 (1);

(15) $c^2(ca)^2c^2 = c^{-1}(ac)^2c$;

(16) $a^{-1}(ca)^2a = a^{-1}(ca)^2a$;

(17) $b^{-1}(ca)^2b = (ac)^2$ *by Lemma* 4 (4);

(18) $c^{-1}(ca)^2c = (ac)^2$;

(19) $aa(ac)^2a^{-1}a^{-1} = a^{-1}(ca)^2a$;

(20) $ba(ac)^2a^{-1}b^{-1} = (ac)^2$ *by Lemma* 4 (4), (1);

(21) $ca(ac)^2a^{-1}c^{-1} = a^{-1}c^{-1}(ac)^2ca$ *by Lemma* 4 (16);

(22) $a^2a(ac)^2a^{-1}a^2 = (ca)^2$;

(23) $b^2a(ac)^2a^{-1}b^2 = (ca)^2$ *by Lemma* 4 (1);

(24) $c^2a(ac)^2a^{-1}c^2 = c^{-1}a^{-1}(ca)^2ac$ *by Lemma* 4 (9), *Lemma* 2 (2);

(25) $a^{-1}a(ac)^2a^{-1}a = (ac)^2$;

(26) $b^{-1}a(ac)^2a^{-1}b = a^{-1}(ca)^2a$ *by Lemma* 4 (7), (10), *Lemma* 2 (2);

(27) $c^{-1}a(ac)^2a^{-1}c = a^{-1}(ca)^2a$ *by Lemma* 4 (9);

(28) $aa^{-1}(ca)^2aa^{-1} = (ca)^2$;

(29) $ba^{-1}(ca)^2ab^{-1} = a(ac)^2a^{-1}$ *by Lemma* 4 (7), (10), *Lemma* 2 (2);

(30) $ca^{-1}(ca)^2ac^{-1} = a(ac)^2a^{-1}$ *by Lemma* 4 (9);

(31) $a^2a^{-1}(ca)^2aa^2 = (ac)^2$;

(32) $b^2a^{-1}(ca)^2ab^2 = (ac)^2$ *by Lemma* 4 (4), (5), (7), (9);

(33) $c^2a^{-1}(ca)^2ac^2 = a^{-1}c^{-1}(ac)^2ca$ *by Lemma* 4 (9), (16);

(34) $a^{-1}a^{-1}(ca)^2aa = a(ac)^2a^{-1}$;

(35) $b^{-1}a^{-1}(ca)^2ab = (ca)^2$ *by Lemma* 4 (5);

(36) $c^{-1}a^{-1}(ca)^2ac = c^{-1}a^{-1}(ca)^2ac$;

(37) $ac(ca)^2c^{-1}a^{-1} = c^{-1}a^{-1}(ca)^2ac$ *by Lemma* 4 (16);

(38) $bc(ca)^2c^{-1}b^{-1} = c^{-1}(ac)^2c$ *by Lemma* 4 (3), (9), (12);

(39) $cc(ca)^2c^{-1}c^{-1} = c^{-1}(ac)^2c$;

(40) $a^2c(ca)^2c^{-1}a^2 = a^{-1}c^{-1}(ac)^2ca$ *by Lemma* 4 (9);

(41) $b^2c(ca)^2c^{-1}b^2 = a^{-1}c^{-1}(ac)^2ca$ *by Lemma* 4 (1), (4), (5), (9);

(42) $c^2c(ca)^2c^{-1}c^2 = (ac)^2$;

(43) $a^{-1}c(ca)^2c^{-1}a = c^{-1}(ac)^2c$ *by Lemma* 4 (9);

(44) $b^{-1}c(ca)^2c^{-1}b = c^{-1}a^{-1}(ca)^2ac$ *by Lemma* 4 (21);

(45) $c^{-1}c(ca)^2c^{-1}c = (ca)^2$;

(46) $ac^{-1}(ac)^2ca^{-1} = c(ca)^2c^{-1}$ *by Lemma* 4 (9);

(47) $bc^{-1}(ac)^2cb^{-1} = a^{-1}c^{-1}(ac)^2ca$ *by Lemma* 4 (18);

(48) $cc^{-1}(ac)^2cc^{-1} = (ac)^2$;

(49) $a^2c^{-1}(ac)^2ca^2 = c^{-1}a^{-1}(ca)^2ac$ *by Lemma* 4 (9), (16);

(50) $b^2c^{-1}(ac)^2cb^2 = c^{-1}a^{-1}(ca)^2ac$ *by Lemma* 4 (18), (20);

(51) $c^2c^{-1}(ac)^2cc^2 = (ca)^2$;

(52) $a^{-1}c^{-1}(ac)^2ca = a^{-1}c^{-1}(ac)^2ca$;

(53) $b^{-1}c^{-1}(ac)^2cb = c(ca)^2c^{-1}$ *by Lemma* 4 (12), (3), (9);

(54) $c^{-1}c^{-1}(ac)^2cc = c(ca)^2c^{-1}$;

(55) $aa^{-1}c^{-1}(ac)^2caa^{-1} = c^{-1}(ac)^2c$;

(56) $ba^{-1}c^{-1}(ac)^2cab^{-1} = c^{-1}a^{-1}(ca)^2ac$ *by Lemma* 4 (20);

(57) $ca^{-1}c^{-1}(ac)^2cac^{-1} = c^{-1}a^{-1}(ca)^2ac$ *by Lemma* 4 (16), (9);

(58) $a^2a^{-1}c^{-1}(ac)^2caa^2 = c(ca)^2c^{-1}$ *by Lemma* 4 (9);

(59) $b^2a^{-1}c^{-1}(ac)^2cab^2 = c(ca)^2c^{-1}$ *by Lemma* 4 (1), (4), (5), (9);

(60) $c^2a^{-1}c^{-1}(ac)^2cac^2 = a^{-1}(ca)^2a$ *by Lemma* 4 (16), (9);

(61) $a^{-1}a^{-1}c^{-1}(ac)^2caa = c^{-1}a^{-1}(ca)^2ac$;

(62) $b^{-1}a^{-1}c^{-1}(ac)^2cab = c^{-1}(ac)^2c$ *by Lemma* 4 (18);

(63) $c^{-1}a^{-1}c^{-1}(ac)^2cac = a(ac)^2a^{-1}$ *by Lemma* 4 (16);

(64) $ac^{-1}a^{-1}(ca)^2aca^{-1} = a^{-1}c^{-1}(ac)^2ca$ *by Lemma* 4 (16), (9);

(65) $bc^{-1}a^{-1}(ca)^2acb^{-1} = c(ca)^2c^{-1}$ *by Lemma* 4 (21);

(66) $cc^{-1}a^{-1}(ca)^2acc^{-1} = a^{-1}(ca)^2a$;

(67) $a^2c^{-1}a^{-1}(ca)^2aca^2 = c^{-1}(ac)^2c$ *by Lemma* 4 (16), (9);

(68) $b^2c^{-1}a^{-1}(ca)^2acb^2 = c^{-1}(ac)^2c$ *by Lemma* 4 (18), (20);

(69) $c^2c^{-1}a^{-1}(ca)^2acc^2 = a(ac)^2a^{-1}$ *by Lemma* 4 (9);

(70) $a^{-1}c^{-1}a^{-1}(ca)^2aca = c(ca)^2c^{-1}$ *by Lemma* 4 (16);

(71) $b^{-1}c^{-1}a^{-1}(ca)^2acb = a^{-1}c^{-1}(ac)^2ca$ *by Lemma* 4 (20);

(72) $c^{-1}c^{-1}a^{-1}(ca)^2acc = a^{-1}c^{-1}(ac)^2ca$ *by Lemma* 4 (9), (16).

The formulae in Lemma 5 show that the elements $(ac)^2$, $(ca)^2$, $a(ac)^2a^{-1}$, $a^{-1}(ca)^2a$, $c(ca)^2c^{-1}$, $c^{-1}(ac)^2c$, $a^{-1}c^{-1}(ac)^2ca$, $c^{-1}a^{-1}(ca)^2ac$ form a full class of conjugates in $H = \langle a, b, c\rangle$. Moreover, the formulae show that they commute pairwise. Hence it is shown that $N_1 = \langle\langle (ac)^2\rangle\rangle_H$ is an elementary abelian normal subgroup of exponent 2 and of order 2^8 in $H = \langle a, b, c\rangle$. It is also shown that the elements $(ac^{-1})^2$ and $(c^{-1}a)^2$ commute elementwise with N.

LEMMA 6. *We have the following rules of conjugation:*

(1) $a(ac^{-1})^2a^{-1} = (a^{-1}c)^2 \cdot a^{-1}(ca)^2a \cdot (ac)^2$ *by Lemma* 5;

(2) $b(ac^{-1})^2b^{-1} = (c^{-1}a)^2$ *by Lemma* 4 (3);

(3) $c(ac^{-1})^2c^{-1} = (a^{-1}c)^2(ca)^2c^{-1}(ac)^2c$ *by Lemma* 5;

(4) $a^2(ac^{-1})^2a^2 = (ca^{-1})^2(ca)^2a(ac)^2a^{-1}$ *by Lemma* 5;

(5) $b^2(ac^{-1})^2b^2 = (ca^{-1})^2(ca)^2a(ac)^2a^{-1}$ *by Lemma* 4 (3), (7), *Lemma* 5;

(6) $c^2(ac^{-1})^2c^2 = (ca^{-1})^2(ac)^2c(ca)^2c^{-1}$ *by Lemma* 5;

(7) $a^{-1}(ac^{-1})^2a = (c^{-1}a)^2$;

(8) $b^{-1}(ac^{-1})^2b = (a^{-1}c)^2 \cdot a^{-1}(ca)^2a \cdot (ac)^2$ *by Lemma* 4 (7), *Lemma* 5;

(9) $c^{-1}(ac^{-1})^2c = (c^{-1}a)^2$;

(10) $a(c^{-1}a)^2a^{-1} = (ac^{-1})^2$;

(11) $b(c^{-1}a)^2b^{-1} = (ca^{-1})^2 \cdot (ca)^2 \cdot a(ac)^2a^{-1}$ *by Lemma* 4 (17);

(12) $c(c^{-1}a)^2c^{-1} = (ac^{-1})^2$;

(13) $a^2(c^{-1}a)^2a^2 = (a^{-1}c)^2a^{-1}(ca)^2a(ac)^2$ *by Lemma* 5;

(14) $b^2(c^{-1}a)^2b^2 = (a^{-1}c)^2a^{-1}(ca)^2a(ac)^2$ *by Lemma* 4 (3), (7), *Lemma* 5;

(15) $c^2(c^{-1}a)^2c^2 = (a^{-1}c)^2(ca)^2c^{-1}(ac)^2c$ *by Lemma* 5;

(16) $a^{-1}(c^{-1}a)^2a = (ca^{-1})^2(ca)^2a(ac)^2a^{-1}$ *by Lemma* 5;

(17) $b^{-1}(c^{-1}a)^2b = (ac^{-1})^2$ *by Lemma* 4 (3);

(18) $c^{-1}(c^{-1}a)^2c = (ca^{-1})^2c(ca)^2c^{-1}(ac)^2$ *by Lemma* 5.

LEMMA 7. *The following elements form a conjugacy class of involutions in* $\langle a, b, c\rangle$ *:*

$$(abcb^{-1})^2,\ (b^{-1}abc)^2,\ a^{-1}(cb^{-1}ab)^2a,\ (cb^{-1}ab)^2,\ a^{-1}(b^{-1}abc)^2a,\ a(abcb^{-1})^2a^{-1},$$
$$a^2(cb^{-1}ab)^2a^2,\ a(cb^{-1}ab)^2a^{-1}\ .$$

Rules of conjugation: (see Proof for comments)

(1) $a(abcb^{-1})^2a^{-1} = a(abcb^{-1})^2a^{-1}$;

(2) $b(abcb^{-1})^2b^{-1} = a^{-1}(cb^{-1}ab)^2a$ *by* p *Lemma* 5 (8), (9) p^{-1} ;

(3) $c(abcb^{-1})^2c^{-1} = (b^{-1}abc)^2$ *by* p *Lemma* 5 (2) p^{-1} ;

(4) $a^2(abcb^{-1})^2a^2 = a^{-1}(b^{-1}abc)^2a$;

(5) $b^2(abcb^{-1})^2b^2 = (cb^{-1}ab)^2$;

(6) $c^2(abcb^{-1})^2c^2 = (cb^{-1}ab)^2$ *by* (3);

(7) $a^{-1}(abcb^{-1})^2a = (b^{-1}abc)^2$;

(8) $b^{-1}(abcb^{-1})^2b = (b^{-1}abc)^2$;

(9) $c^{-1}(abcb^{-1})^2c = a^{-1}(cb^{-1}ab)^2a$ *by* p *Lemma* 5 (1), (8) p^{-1} ;

(10) $a(b^{-1}abc)^2a^{-1} = (abcb^{-1})^2$ *by* (*);

(11) $b(b^{-1}abc)^2b^{-1} = (abcb^{-1})^2$;

(12) $c(b^{-1}abc)^2c^{-1} = (cb^{-1}ab)^2$;

(13) $a^2(b^{-1}abc)^2a^2 = a(abcb^{-1})^2a^{-1}$ *by* (10);

(14) $b^2(b^{-1}abc)^2b^2 = a^{-1}(cb^{-1}ab)^2a$ *by* (2);

(15) $c^2(b^{-1}abc)^2c^2 = a^{-1}(cb^{-1}ab)^2a$ *by* p *Lemma* 5 (13), (14) p^{-1} ;

(16) $a^{-1}(b^{-1}abc)^2a = a^{-1}(b^{-1}abc)^2a$;

(17) $b^{-1}(b^{-1}abc)^2b = (cb^{-1}ab)^2$ *by* (*);

(18) $c^{-1}(b^{-1}abc)^2c = (abcb^{-1})^2$ *by* (3);

(19) $aa^{-1}(cb^{-1}ab)^2aa^{-1} = (cb^{-1}ab)^2$;

(20) $ba^{-1}(cb^{-1}ab)^2ab^{-1} = (cb^{-1}ab)^2$ *by* (2), (5);

(21) $ca^{-1}(cb^{-1}ab)^2ac^{-1} = (abcb^{-1})^2$ *by* p *Lemma* 5 (17), (53), (43) p^{-1} ;

(22) $a^2a^{-1}(cb^{-1}ab)^2aa^2 = a(cb^{-1}ab)^2a^{-1}$;

(23) $b^2a^{-1}(cb^{-1}ab)^2ab^2 = (b^{-1}abc)^2$ *by* (*);

(24) $c^2a^{-1}(cb^{-1}ab)^2ac^2 = (b^{-1}abc)^2$ *by* (3), (9);

(25) $a^{-1}a^{-1}(cb^{-1}ab)^2aa = a^2(cb^{-1}ab)^2a^2$;

(26) $b^{-1}a^{-1}(cb^{-1}ab)^2ab = (abcb^{-1})^2$ *by* (2);

(27) $c^{-1}a^{-1}(cb^{-1}ab)^2ac = (cb^{-1}ab)^2$ *by* (24), (12);

(28) $a(cb^{-1}ab)^2a^{-1} = a(cb^{-1}ab)^2a^{-1}$;

(29) $b(cb^{-1}ab)^2b^{-1} = (b^{-1}abc)^2$ *by* (17);

(30) $c(cb^{-1}ab)^2c^{-1} = a^{-1}(cb^{-1}ab)^2a$ *by* (27);

(31) $a^2(cb^{-1}ab)^2a^2 = a^2(cb^{-1}ab)^2a^2$;

(32) $b^2(cb^{-1}ab)^2b^2 = (abcb^{-1})^2$ *by* (5);

(33) $c^2(cb^{-1}ab)^2c^2 = (abcb^{-1})^2$ *by* (3);

(34) $a^{-1}(cb^{-1}ab)^2a = a^{-1}(cb^{-1}ab)^2a$;

(35) $b^{-1}(cb^{-1}ab)^2b = a^{-1}(cb^{-1}ab)^2a$ *by* (20);

(36) $c^{-1}(cb^{-1}ab)^2c = (b^{-1}abc)^2$ *by* (12);

(37) $aa^{-1}(b^{-1}abc)^2aa^{-1} = (b^{-1}abc)^2$;

(38) $ba^{-1}(b^{-1}abc)^2ab^{-1} = a(cb^{-1}ab)^2a^{-1}$ *by* p *Lemma* 5 (17), (47), (63), (20) p^{-1} ;

(39) $ca^{-1}(b^{-1}abc)^2ac^{-1} = a(abcb^{-1})^2a^{-1}$ *by* p *Lemma* 5 (8), (27), (35) p^{-1} ;

(40) $a^2a^{-1}(b^{-1}abc)^2aa^2 = (abcb^{-1})^2$ *by* (10);

(41) $b^2a^{-1}(b^{-1}abc)^2ab^2 = a^2(cb^{-1}ab)^2a^2$ *by* p *Lemma* 5 (8), (24), (68), (2) p^{-1} ;

(42) $c^2a^{-1}(b^{-1}abc)^2ac^2 = a^2(cb^{-1}ab)^2a^2$ *by* p *Lemma* 5 (8), (27), (34), (26), (68), (2) p^{-1} ;

(43) $a^{-1}a^{-1}(b^{-1}abc)^2aa = a(abcb^{-1})^2a^{-1}$ *by* (13);

(44) $b^{-1}a^{-1}(b^{-1}abc)^2ab = a(abcb^{-1})^2a^{-1}$ *by* p *Lemma* 5 (8), (27), (35) p^{-1} ;

(45) $c^{-1}a^{-1}(b^{-1}abc)^2ac = a(cb^{-1}ab)^2a^{-1}$ *by* p *Lemma* 5 (8), (21), (56), (64), (62), (2) p^{-1} ;

(46) $aa(abcb^{-1})^2a^{-1}a^{-1} = a^{-1}(b^{-1}abc)^2a$ *by* (4);

(47) $ba(abcb^{-1})^2a^{-1}b^{-1} = a^{-1}(b^{-1}abc)^2a$ *by* (44);

(48) $ca(abcb^{-1})^2a^{-1}c^{-1} = a^2(cb^{-1}ab)^2a^2$ *by* p *Lemma* 5 (11), (34), (26), (68), (2) p^{-1} ;

(48) $a^2a(abcb^{-1})^2a^{-1}a^2 = (b^{-1}abc)^2$ *by* (7);

(50) $b^2a(abcb^{-1})^2a^{-1}b^2 = a(cb^{-1}ab)^2a^{-1}$ *by* p *Lemma* 5 (11), (33), (62), (2) p^{-1};

(51) $c^2a(abcb^{-1})^2a^{-1}c^2 = a(cb^{-1}ab)^2a^{-1}$ *by* p *Lemma* 5 (11), (30), (19), (29), (21), (62), (2) p^{-1} ;

(52) $a^{-1}a(abcb^{-1})^2a^{-1}a = (abcb^{-1})^2$;

(53) $b^{-1}a(abcb^{-1})^2a^{-1}b = a^2(cb^{-1}ab)^2a^2$ *by* p *Lemma* 5 (11), (68), (2) p^{-1} ;

(54) $c^{-1}a(abcb^{-1})^2a^{-1}c = a^{-1}(b^{-1}abc)^2a$ *by* (34);

(55) $aa^2(cb^{-1}ab)^2a^2a^{-1} = a^{-1}(cb^{-1}ab)^2a$;

(56) $ba^2(cb^{-1}ab)^2a^2b^{-1} = a(abcb^{-1})^2a^{-1}$ *by* (53);

(57) $ca^2(cb^{-1}ab)^2a^2c^{-1} = a(cb^{-1}ab)^2a^{-1}$ *by* p *Lemma* 5 (17), (50), (70), (44), (72), (62), (2), p^{-1} ;

(58) $a^2a^2(cb^{-1}ab)^2a^2a^2 = (cb^{-1}ab)^2$;

(59) $b^2a^2(cb^{-1}ab)^2a^2b^2 = a^{-1}(b^{-1}abc)^2a$ *by* (41);

(60) $c^2a^2(cb^{-1}ab)^2a^2c^2 = a^{-1}(b^{-1}abc)^2a$ *by* (42);

(61) $a^{-1}a^2(cb^{-1}ab)^2a^2a = a(cb^{-1}ab)^2a^{-1}$;

(62) $b^{-1}a^2(cb^{-1}ab)^2a^2b = a(cb^{-1}ab)^2a^{-1}$ *by* p *Lemma* 5 (17), (50), (72), (62), (2), p^{-1} ;

(63) $c^{-1}a^2(cb^{-1}ab)^2a^2c = a(abcb^{-1})a^{-1}$ *by* (48);

(64) $aa(cb^{-1}ab)^2a^{-1}a^{-1} = a^2(cb^{-1}ab)^2a^2$;

(65) $ba(cb^{-1}ab)^2a^{-1}b^{-1} = a^2(cb^{-1}ab)^2a^2$ *by* (62);

(66) $ca(cb^{-1}ab)^2a^{-1}c^{-1} = a^{-1}(b^{-1}abc)^2a$ *by* (45);

(67) $a^2a(cb^{-1}ab)^2a^{-1}a^2 = a^{-1}(cb^{-1}ab)^2a$;

(68) $b^2a(cb^{-1}ab)^2a^{-1}b^2 = a(abcb^{-1})^2a^{-1}$ *by* (50);

(69) $c^2a(cb^{-1}ab)^2a^{-1}c^2 = a(abcb^{-1})^2a^{-1}$ *by* (51);

(70) $a^{-1}a(cb^{-1}ab)^2a^{-1}a = (cb^{-1}ab)^2$;

(71) $b^{-1}a(cb^{-1}ab)^2a^{-1}b = a^{-1}(b^{-1}abc)^2a$ *by* (38);

(72) $c^{-1}a(cb^{-1}ab)^2a^{-1}c = a^2(cb^{-1}ab)^2a^2$ *by* (57).

Proof. The term p Lemma $x(y)p^{-1}$ on the right-hand side means that the appropriate formula stated on the left-hand side is implied by applying the automorphism induced by p on $\langle a, b, c\rangle$ on the formula Lemma $x(y)$.

By Lemma 5 we have

$$ab(ca)^2b^{-1}a^{-1} = (ca)^2 .$$

Applying p , we obtain

(*) $$(bcb^{-1}a)^2 = (b^{-1}abc)^2$$

which yields (4), (5), (7).

LEMMA 8. *The following rules of transformation apply:*

(1) $a(abcb)^2a^{-1} = (c^{-1}b^{-1}a^{-1}b^{-1})^2a^{-1}(b^{-1}abc)^2a\ (abcb^{-1})^2$;

(2) $b\ (abcb)^2b^{-1} = (babc)^2$;

(3) $c(abcb)^2c^{-1} = (c^{-1}b^{-1}a^{-1}b^{-1})^2\ (b^{-1}abc)^2\ a^{-1}(cb^{-1}ab)^2a$;

(4) $a^2(abcb)^2a^{-2} = (b^{-1}c^{-1}b^{-1}a^{-1})^2\ (b^{-1}abc)^2\ a(abcb^{-1})^2a^{-1}$;

(5) $b^2(abcb)^2b^{-2} = (b^{-1}c^{-1}b^{-1}a^{-1})^2\ (abcb^{-1})^2\ (cb^{-1}ab)^2$;

(6) $c^2(abcb)^2c^{-2} = (b^{-1}c^{-1}b^{-1}a^{-1})^2\ (abcb^{-1})^2\ (cb^{-1}ab)^2$;

(7) $a^{-1}(abcb)^2a = (babc)^2$;

(8) $b^{-1}(abcb)^2b = (c^{-1}b^{-1}a^{-1}b^{-1})^2(b^{-1}abc)^2a^{-1}(cb^{-1}ab)^2a$;

(9) $c^{-1}(abcb)^2c = (babc)^2$;

(10) $a\cdot(babc)^2a^{-1} = (abcb)^2$;

(11) $b(babc)^2b^{-1} = (b^{-1}c^{-1}b^{-1}a^{-1})^2(cb^{-1}ab)^2\ (abcb^{-1})^2$;

(12) $c(babc)^2c^{-1} = (abcb)^2$;

(13) $a^2(babc)^2a^{-2} = (c^{-1}b^{-1}a^{-1}b^{-1})^2a^{-1}(b^{-1}abc)^2a\ (abcb^{-1})^2$;

(14) $b^2(babc)^2 b^{-2} = (c^{-1}b^{-1}a^{-1}b^{-1})^2 (b^{-1}abc)^2 a^{-1}(cb^{-1}ab)^2 a$;

(15) $c^2(babc)^2 c^{-2} = (c^{-1}b^{-1}a^{-1}b^{-1})^2 (b^{-1}abc)^2 a^{-1}(cb^{-1}ab)^2 a$;

(16) $a^{-1}(babc)^2 a = (b^{-1}c^{-1}b^{-1}a^{-1})^2 (b^{-1}abc)^2 a(abcb^{-1})^2 a^{-1}$;

(17) $b^{-1}(babc)^2 b = (abcb)^2$;

(18) $c^{-1}(babc)^2 c = (b^{-1}c^{-1}b^{-1}a^{-1})^2 (cb^{-1}ab)^2 (abcb^{-1})^2$.

Proof. Transform the formulae of Lemma 6 by p and use Lemma 7 and Lemma 1 to conclude Lemma 8. Transform the normal subgroup generated by $(ac)^2$ in H by p to obtain the normal subgroup generated by $(abcb^{-1})^2$. Hence this subgroup is also elementary abelian. The formulae of Lemma 5 imply that $(abcb^{-1})^2$ commutes with all eight conjugates of $(ac)^2$. Hence

$$N := \langle (ac)^2, (ca)^2, a(ac)^2a^{-1}, a^{-1}(ca)^2a, c(ca)^2c^{-1}, c^{-1}(ac)^2c, a^{-1}c^{-1}(ac)^2ca, c^{-1}a^{-1}(ca)^2ac, (abcb^{-1})^2, (b^{-1}abc)^2, a^{-1}(cb^{-1}ab)^2a, (cb^{-1}ab)^2, a^{-1}(b^{-1}abc)^2a, a(abcb^{-1})^2a^{-1}, a^2(cb^{-1}ab)^2a^2, a(cb^{-1}ab)^2a^{-1}\rangle$$

is an elementary abelian normal subgroup in H . The formulae of Lemma 5, Lemma 7 show that $(c^{-1}a)^2$ and $(ac^{-1})^2$ centralize the group N . Transforming this statement by p implies that $(abcb)^2$ and $(babc)^2$ centralize N .

Moreover Lemma 6 yields the commutator relation

$$(a^{-1}c)^2\cdot(ac^{-1})^2\cdot(c^{-1}a)^2\cdot(ca^{-1})^2 = (ac)^2\cdot(ca)^2\cdot a(ac)^2a^{-1}\cdot a^{-1}(ac)^2a\cdot c^{-1}(ac)^2c \cdot c(ca)^2c^{-1}\cdot a^{-1}c^{-1}(ac)^2ca\cdot c^{-1}a^{-1}(ca)^2ac \ .$$

Transform by p obtaining

$$(abcb)^{-2}(babc)^2(abcb)^2 (babc)^{-2} = (abcb^{-1})^2\cdot(b^{-1}abc)^2\cdot a^{-1}(cb^{-1}ab)^2a\cdot(cb^{-1}ab)^2\cdot a^{-1}(b^{-1}abc)^2a\cdot a(abcb^{-1})^2a^{-1} \cdot a^2(cb^{-1}ab)^2a^{-2}a(cb^{-1}ab)^2a^{-1} \ .$$

LEMMA 9. *In* $\langle a, b, c, F\rangle$ *we have the following relations:*

(1) $ac^2b^{-1}a^{-1}b^{-1}ca^2bc^{-1}b = c^{-1}a^{-1}(ca)^2ac\cdot(ac)^2\cdot c(ca)^2c^{-1} \cdot a^{-1}c^{-1}(ac)^2ca(ca)^2a^{-1}(ca)^2a$;

(2) $b^{-1}a^{-1}b^{-1}cba^2cba = (ac)^2c^{-1}a^{-1}(ca)^2ac$;

(3) $(c^{-1}b^{-1})^2a^{-1}c^{-1}(bc)^2ca = (ca)^2c^{-1}(ac)^2c$;

(4) $(bc^2a)^2 = (a^{-1}c^{-1}bc)^2$;

(5) $(ab^2a^{-1}b^2)^2 = (bc^2b^{-1}c^2)^2 = 1$;

(6) $(abc^{-1}b)^4 = 1$;

(7) $(bab^2abc^{-1}a)^2 = (ac)^2a^{-1}(ca)^2aa^{-1}c^{-1}(ac)^2ca$;

(8) $(bcabc^{-1}b^2a^{-1})^2 = 1$;

(9) $b\cdot a^2b^2a^2\cdot b^{-1}\cdot a^2b^2a^2 = a^{-1}c^{-1}(ac)^2ca\cdot c(ca)^2c^{-1}\cdot a(ac)^2a^{-1}\cdot (ca)^2$;

(10) $(babcbaba^{-1})^2 = (ac)^2a^{-1}(ca)^2aa^{-1}c^{-1}(ac)^2ca$;

(11) $(abca^{-1}b^{-1}c^{-1})^2 = 1$;

(12) $(abcb^{-1})^2a^{-1}(b^{-1}abc)^2a = a^{-1}(ca)^2aa^{-1}c^{-1}(ac)^2ca$;

(13) $(bcbc^{-1})^2 = (b^{-1}abc)^2\cdot a^{-1}(cb^{-1}ab)^2a(c^{-1}b^{-1}a^{-1}b^{-1})^2\cdot (c^{-1}a)^2$;

(14) $(abab^{-1})^2 = (ca)^2a(ac)^2a^{-1}(ca^{-1})^2(b^{-1}c^{-1}b^{-1}a^{-1})^2$;

(15) $Fbcb^{-1}aF = a^{-1}(ca)^2a\cdot c(ca)^2c^{-1}\cdot b^{-1}ca^2ba^{-1}$;

(16) $Fab^{-1}cb^{-1}F = a^{-1}c^{-1}abca^2bca^{-1}$;

(17) $Fbc^{-1}bcF = a(ac)^2a^{-1}c(ca)^2c^{-1}c^{-1}a^{-1}(ca)^2aca^{-1}c^{-1}(ac)^2ca$
$a(cb^{-1}ab)^2a^{-1}a(abcb^{-1})^2a^{-1}(a^{-1}c)^4(c^{-1}b^{-1}a^{-1}b^{-1})^2\cdot ac^{-1}b^{-1}a^{-1}b^{-1}c$;

(18) $FacF = (ca)^2(ac)^2a^{-1}c^{-1}(ac)^2cac^{-1}(ac)^2c(a^{-1}c)^4(c^{-1}b^{-1}a^{-1}b^{-1})^2$
$a^{-1}c^{-1}abcab^{-1}c$;

(19) $Fabcb^{-1}F = (ac)^2a^{-1}c^{-1}(ac)^2ca\cdot a^{-1}b^{-1}a^2c^{-1}b^2c^2b^{-1}$;

(20) $Fab^2cF = a(ac)^2a^{-1}c^{-1}a^{-1}(ca)^2ac(b^{-1}abc)^2a^{-1}(cb^{-1}ab)^2a(a^{-1}c)^4(babc)^2$
$\cdot a^{-1}b^{-1}a^2c^{-1}b^2c^2b^{-1}ac^{-1}b^{-1}a^{-1}b^{-1}c$;

(21) $c^{-1}a^2ba^{-1}cb^{-1}a^2c^{-1}b^2c^2b^{-1}ac^{-1}b^{-1} = a(ac)^2a^{-1}c^{-1}(ac)^2c$;

(22) $FcaF = a(ac)^2a^{-1}a^{-1}(ca)^2ac(ca)^2c^{-1}c^{-1}a^{-1}(ca)^2ac(ca)^2c^{-1}(ac)^2ca^{-1}(cb^{-1}ab)^2a$
$(cb^{-1}ab)^2a^2(cb^{-1}ab)^2a^2a(cb^{-1}ab)^2a^{-1}(c^{-1}a)^4(c^{-1}b^{-1}a^{-1}b^{-1})^2abcab^{-1}a^{-1}$.

Proof. (1) By Lemma 2 (12) we have

$$abc^{-1}b^{-1}a^{-1}D^2D^{-1}bc^{-1}bca^{-1}b^{-1}DD^2b^{-1}ca = 1 .$$

Lemma 2 (5), Lemma 3 (2), (8), (1) and Lemma 4 (1) imply

$$c^{-1}b^{-1}a^2c^{-1}a^2ca^{-1}b^{-1}ac^2ac^{-1}a^2caca^{-1}bc^{-1}bca^{-1} = 1 .$$

Hence

$$bc^{-1}bac^2\cdot c(ca^{-1})^2c^{-1}\cdot b^{-1}a^{-1}\cdot(a^{-1}c^{-1})^2(ca^{-1})^2\cdot b^{-1}\cdot ac^2ac^{-1}a^2caca^{-1} = 1 .$$

Lemmas 5 and 6 yield (1).

(2) By Lemma 2 (11) we have

$$b^{-1}D^{-1}b^{-1}cb^{-1}D^{-1}abca^{-1}Dbc^{-1}bDbac^{-1}b^{-1}a^{-1} = 1 .$$

Lemma 3 (4), (7), (1), Lemma 4 (1) and Lemma 5 imply

$$b^{-1}D^{-1}b^{-1}ac^{-1}ab^2Dbac^{-1}b^{-1}a^{-1} = 1 .$$

Lemma 3 (4), (1), (3), (7), (2) and Lemma 4 (1) imply

$$b^{-1}a^{-1}c^{-1}b^{-1}a^2c^{-1}ac^{-1}a^2b^{-1}c^2a^{-1}ca^{-1}b\cdot bab\cdot cac^{-1} = 1 .$$

Use Lemma 9 (1) to replace bab and apply Lemmas 5 and 6 obtaining Lemma 9 (2).

(3) By Lemma 2 (25), (26) we have

$$c^{-1}bcbca^{-1}c^2b^{-1}a^2\cdot b^{-1}D^{-1}Fbca^{-1}Fa^{-1}c^{-1}Fac^{-1}\cdot b^{-1}FDba\cdot c^2b^{-1}a^{-1} = 1 .$$

By Lemma 2 (27),

$$c^{-1}bcbca^{-1}c^2b^{-1}a^{-1}c^{-1}a^{-1}c^{-1}c^{-1}b^{-1}a^{-1} = 1 .$$

Hence Lemma 5 yields (3).

(4) By Lemma 2 (25), (28) we have

$$bc^2a^{-1}b^{-1}D^{-1}Fbca^{-1}F\cdot bc^2a^{-1}b^{-1}D^{-1}Fbca^{-1}F = a^{-1}c^{-1}bca^{-1}c^{-1}bc .$$

Hence Lemma 2 (26) implies (4).

(5) By Lemma 2 (13) we have

$$(D^{-1}abca^{-1}Db^{-1}c^{-1})^2 = 1 .$$

By Lemma 2 (5), Lemma 3 (2), (4) we have

$$(ab^2cab^2ac^2b^2a^{-1}b^2c^{-1})^2 = 1 .$$

Using Lemma 4 (1) we have

$$(a^{-1}c^{-1}b^2a^{-1}b^2c^{-1})^2 = 1 .$$

Then Lemma 5 implies the first part of (5). When we apply to $(ab^2a^{-1}b^2)^2 = 1$ the automorphism induced by p on $\langle a, b, c\rangle$, we obtain the second part of (5).

(6) Lemma 2 (14) yields

$$\left(D^2D^{-1}ba^2bDba^2b\right)^2 = 1 .$$

Lemma 3 (10), (5), (4), Lemma 4 (1) imply

$$\left(abca^2b\cdot caca\cdot abca^2b\right)^2 = 1 .$$

Use Lemma 5 to conclude

$$\left(abca^2b\right)^4 = 1 .$$

Apply to this equation the automorphism induced by p on $\langle a, b, c\rangle$ obtaining (6).

(7) By Lemma 2 (15) we have

$$\left(Dba^2bc^{-1}bc^{-1}b\right)^2 = 1 .$$

Lemma 3 (10), (5), (4), (8), (2), Lemma 4 (1) imply

$$babca^2bcac^2a^{-1}bac^{-1}a^2c^2a^{-1}bac^{-1}a^2ca^2bca^2bc^{-1}bc^{-1} = 1 .$$

From Lemma 5 follows

$$\left(bab\cdot ca^2bc^{-1}bc^{-1}\right)^2 = (ac)^2(ca)^2a^{-1}(ca)^2aa^{-1}c^{-1}(ac)^2cac^{-1}a^{-1}(ca)^2ac .$$

Replacing $ca^2bc^{-1}b$ by Lemma 9 (1) and using Lemma 5 yields (7).

(8) and (11) By the definition of a, b, c we have

$$bcabc^{-1}b^2a^{-1} = \left(pqp^2q^{-1}\right)^4 ,$$

$$abca^{-1}b^{-1}c^{-1} = \left(qp^{-1}q^{-1}p^2\right)^4 .$$

Then (xvi) and (xvii), respectively, imply (8) and (11), respectively.

(9) By Lemma 2 (17) we have

$$ac^{-1}D^{-1}abcac^{-1}D^{-1}abcb^{-1}\cdot b^2\cdot bc^{-1}b^{-1}a^{-1}Dca^{-1}c^{-1}b^{-1}a^{-1}Dca^{-1}b^2 = 1 .$$

Lemma 2 (5), Lemma 3 (2) imply

$$ac^{-1}D^{-1}abcb^{-1}bac^{-1}a^{-1}c^{-1}b^2caca^{-1}c^{-1}b^{-1}a^{-1}Dca^{-1}b^2 = 1 .$$

By Lemma 2 (5), Lemma 3 (5), (6), (2), (8), Lemma 4 (1), Lemma 5 follows

$$\begin{aligned}&(ca)^2\cdot a(ac)^2a^{-1}b\cdot(ca)^2a^2(ac)^2a(ac)^2a^{-1}c(ca)^2c^{-1}\\ &\qquad c^{-1}a^{-1}(ca)^2aca^{-1}c^{-1}(ac)^2cab^2ac^{-1}a^2\left(a^{-1}c\right)^2ac^{-1}b^{-1}ca^{-1}b^2ac^{-1} = 1 .\end{aligned}$$

Applying again Lemmas 5 and 6, implies Lemma 9 (9).

(10) By Lemma 2 (18) we have

$$\left(D^2D^{-1}baDbc^{-1}bDbac^{-1}\right)^2 = 1 .$$

By Lemma 3 (10), (5), Lemma 4 (1), Lemma 5 we have

$$\left(bac^{-1}abca^2b^2D\right)^2 = 1 .$$

By Lemma 3 (10), (5), (3), (7), (8), (1), Lemma 4 (1) we have

$$ac^{-1}abca^2b^2abc^{-1}acab^{-1}a^{-1}c^{-1}b^{-1}ca^2c^{-1}a^2b^{-1}c^{-1}a^2ca^{-1}b^2abc = 1 .$$

Hence

$$ac^{-1}ab\cdot\left(ca^{-1}\right)^2\cdot a(ac)^2a^{-1}\cdot a^2cb^2abc^2\cdot(ca)^2\cdot ba^2c\cdot(ac)^2a^{-1}b\left(ca^{-1}\right)^2a(ac)^2a^{-1}$$
$$\cdot b^{-1}\left(c^{-1}a\right)^2\cdot a^{-1}(ca)^2a\cdot ab^2abc = 1$$

By Lemmas 5 and 6 follows

$$\left(c^2\cdot ba^2cb^2ab\right)^2 = (ca)^2a(ac)^2a^{-1}a^{-1}(ca)^2ac^{-1}(ac)^2c\cdot c(ca)^2c^{-1} .$$

Replace cba^2cb by Lemma 9 (2) and use Lemma 5 to obtain (10).

(12) By Lemma 9 (10) we have

$$(babc)^2c^{-1}a^{-1}(babc)^2c^{-1}a^{-1} = (ac)^2\cdot a^{-1}(ca)^2aa^{-1}c^{-1}(ac)^2ca .$$

Applying Lemma 8 we obtain Lemma 9 (12).

(13) By Lemma 9 (3) follows

$$bcbc^{-1}\cdot\left(c^{-1}a\right)^2\cdot bcb\cdot(abcb)^{-2}\cdot c^{-1} = (ca)^2\cdot c^{-1}(ac)^2c .$$

Use Lemma 5, Lemma 6, Lemma 7, Lemma 8 to conclude Lemma 9 (13).

(14) Apply to Lemma 9 (13) the automorphism induced by p^{-1} on $\langle a, b, c\rangle$ obtaining Lemma 9 (14).

(15) By Lemma 9 (29) we have

$$Dbc^{-1}bD^2\cdot D^{-1}bac^{-1}b^{-1}a^{-1}b^{-1}D\cdot D^2ca^{-1}b^{-1}D^{-1}Fbcb^{-1}aF = 1 .$$

Using Lemma 3 (10), (5), (8), Lemma 4 (1) we obtain

$$D^2D^{-1}bc^{-1}babc^2a^{-1}b^{-1}a^{-1}c^{-1}b^{-1}a^{-1}ca^{-1}b^{-1}DD^2Fbcb^{-1}aF = 1 .$$

By Lemma 3 (10), (8), (3), (9), (4), (5), (7), (1), Lemma 4 (1) follows

$$abcb^{-1}\cdot(ca)^2\cdot bc^{-1}babac^{-1}acb\cdot a(ac)^2a^{-1}\left(ac^{-1}\right)^2$$
$$\cdot b^{-1}a^2c^{-1}a^{-1}c^{-1}bcaba^{-1}c^{-1}a^{-1}cb^{-1}a^{-1}Fbcb^{-1}aF = 1$$

By Lemmas 6 and 7 follows

$$ab\cdot(ca)^2\cdot bab\cdot ac^{-1}a^2c^{-1}a(ac)^2a^{-1}(ca)^2bcaba^{-1}c^{-1}a^{-1}cb^{-1}a^{-1}Fbcb^{-1}aF = 1 .$$

Replace bab using Lemma 9 (1) and apply Lemma 4 (1), Lemmas 6 and 7 to obtain

$$a\cdot ba^2cb\cdot c(ca)^2c^{-1}\cdot (ac)^2c^{-1}(ac)^2cc^{-1}a^{-1}(ca)^2aca^{-1}(ca)^2a(ca)^2\cdot ab^{-1}c^2b^{-1}a^{-1}Fbcb^{-1}aF = 1 \; .$$

Replace ba^2cb using Lemma 9 (2), and apply Lemmas 6 and 7 and replace again using Lemma 9 (1) to obtain (15).

(16) follows immediately by Lemma 2 (23), (33).

(17) By Lemma 2 (31) we have

$$Fc^{-1}b^{-1}cb^{-1}Fac^{-1}b^{-1}a^{-1}c^{-1}bD^2\cdot D^{-1}bac^{-1}b^{-1}a^{-1}b^{-1}DD^2c^{-1}b^{-1} = 1 \; .$$

By Lemma 3 (10), (5), (8), Lemma 4 (1) we have

$$Fbc^{-1}bcF = ac^{-1}b^{-1}a^{-1}c^{-1}ba\cdot bc^2a^{-1}b^{-1}\cdot a^{-1}c^{-1}b^{-1}a^{-1}c^{-1}b^{-1} \; .$$

Replace $bc^2a^{-1}b^{-1}$ using Lemma 9 (1) and apply Lemma 6 to obtain

$$Fbc^{-1}bcF = a(ac)^2a^{-1}a^{-1}(ca)^2ac(ca)^2c^{-1}c^{-1}(ac)^2cc^{-1}a^{-1}(ca)^2ac\cdot ac^{-1}a^{-1}b^{-1}a^{-1} \cdot \left(abab^{-1}\right)^2\cdot a^{-1}c^{-1}b^{-1} \; .$$

Replace $\left(abab^{-1}\right)^2$ using Lemma 9 (14) and apply Lemmas 6, 7, 8, 5 to obtain

$$Fbc^{-1}bcF = c^{-1}(ac)^2ca^{-1}c^{-1}(ac)^2ca(ac)^2\left(a^{-1}c\right)^2a^{-1}\left(b^{-1}abc\right)^2a\left(abcb^{-1}\right)^2\cdot a\left(cb^{-1}ab\right)^2a^{-1} \cdot a\left(abcb^{-1}\right)^2a^{-1}\left(c^{-1}b^{-1}a^{-1}b^{-1}\right)^2ac^{-1}a^{-1}b^{-1}c^{-1}a^2\cdot a(ac)^2a^{-1}\left(ac^{-1}\right)^2 b^{-1} \; .$$

Apply Lemmas 5 and 6, replace $b^{-1}c^{-1}a^2b^{-1}$ using Lemma 9 (2), and apply Lemmas 5 and 6 and Lemma 9 (12) to obtain (17).

(18) follows by Lemma 9 (16), (17) using Lemmas 5 to 8.

(19) follows by Lemma 2 (20), Lemma 9 (15) using Lemma 5.

(20) follows by Lemma 9 (17), (19) using Lemmas 5, 7, 8.

(21) Lemma 2 (24) implies

$$abca^{-1}\cdot Fbc^{-1}bc\cdot c^{-1}a^{-1}\cdot abcb^{-1}F\cdot ac^{-1}b^{-1}a^{-1}b^2 = 1 \; .$$

By Lemma 9 (17), (18), (19), Lemma 4 (1) we have

$$a^{-1}c^{-1}b^{-1}a^{-1}cb^{-1}a^2c^{-1}b^2c^2b^{-1}ac^{-1}b^{-1}a^{-1}b^2 = (ac)^2\cdot a^{-1}c^{-1}(ac)^2ca \; .$$

Lemma 5, Lemma 4 (1) imply (21).

(22) By Lemma 9 (20), Lemma 2 (21) and Lemmas 5 to 8 we have

$$FcaF = \left(c^{-1}a\right)^4\cdot\left(c^{-1}b^{-1}a^{-1}b^{-1}\right)^2a^{-1}\left(cb^{-1}ab\right)^2a\cdot\left(b^{-1}abc\right)^2\left(cb^{-1}ab\right)^2\cdot a^2\left(cb^{-1}ab\right)^2a^2 \cdot a\left(cb^{-1}ab\right)^2a^{-1}a\left(abcb^{-1}\right)^2a^{-1}c^{-1}a^{-1}(ca)^2aca(ac)^2a^{-1}\cdot c^{-1}ba\cdot bca^{-1}bc^2b^2ca^2bc^{-1}\cdot c^{-1}b^2a^{-1} \; .$$

Applying Lemma 9 (21), Lemma 5, Lemma 9 (12) we obtain

$$FcaF = a(ac)^2a^{-1}a^{-1}(ca)^2ac(ca)^2c^{-1}c^{-1}a^{-1}(ca)^2ac(ca)^2c^{-1}(ac)^2ca^{-1}(cb^{-1}ab)^2a(cb^{-1}ab)^2$$
$$a^2(cb^{-1}ab)^2a^2a(cb^{-1}ab)^2a^{-1}\cdot c^{-1}\cdot bac^{-1}a^2b\cdot a^{-1}c^{-1}b^2a^{-1}$$

Then follows (22) by Lemma 4 (1), Lemma 2 (2).

LEMMA 10. *We have*

(1) $q(ac)^2q^{-1} = (ca)^2$;

(2) $q(ca)^2q^{-1} = (ac)^2$;

(3) $(c^{-1}a)^4 = (ca)^2a(ac)^2a^{-1}$, $(ac^{-1})^4 = (ac)^2a^{-1}(ca)^2a$;

(4) $(abcb)^4 = (b^{-1}abc)^2\cdot a^{-1}(cb^{-1}ab)^2a$, $(babc)^4 = (abcb^{-1})^2\cdot(cb^{-1}ab)^2$;

(5) $\langle (ac)^2, (ca)^2\rangle$, $\langle (abcb^{-1})^2, (b^{-1}abc)^2\rangle$, $\langle (abcb)^2, (babc)^2\rangle$, $\langle (c^{-1}a)^2, (ac^{-1})^2\rangle$ *are four elementary abelian normal subgroups of exponent* 2 *in* $\langle a, b, c\rangle$. *They centralize each other elementwise. Each of them is centralized by* H_g . *Any element of* $H\backslash H_g$ *permutes, in each case, the two generators.*

Proof. (1) By Lemma 1 we have

$$qacacq^{-1} = Dbc^{-1}b^{-1}a^{-1}Dbc^{-1}b^{-1}a^{-1} .$$

Lemma 2 (5) and Lemma 3 (10) imply (1).

(2) By Lemma 1 we have

$$qcacaq^{-1} = adbc^{-1}b^{-1}a^{-1}Dbc^{-1}b^{-1}a^{-1}da^{-1} .$$

By Lemma 2 (5) we have

$$qcacaq^{-1} = a(abcb^{-1})^2a^{-1}aD^2a^{-1} .$$

Lemma 5, Lemma 3 (10) imply (2).

(3) By Lemma 2 (30), Lemma 3 (2), (8), (6), (3), (5), (10) and Lemma 4 (1) we have

$$Fbcb^{-1}aFa^{-1}c^{-1}b^{-1}a^2c^{-1}a^2ca^{-1}b^{-1}ac^{-1}acba^{-1}cac^{-1}a^2cacb^2c^{-1}a^{-1}cb^{-1}a^{-1} = 1 .$$

By Lemma 9 (15) we have

$$a^{-1}(ca)^2a\cdot c(ca)^2c^{-1}\cdot b^{-1}ca^2b\cdot a^2c^{-1}b^{-1}a^2c^{-1}a^2ca^{-1}b^{-1}ac^{-1}acba^{-1}cac^{-1}a^2cacb^2$$
$$\cdot c^{-1}a^{-1}cb^{-1}a^{-1} = 1$$

Use Lemma 9 (2) to replace $b^{-1}ca^2b$ obtaining

$$ba^2 \cdot b^{-1}a^{-1}b^{-1} \cdot ac^{-1}acba^{-1}cac^{-1}a \cdot (ac)^2 \cdot (ca)^2\ a(ac)^2a^{-1}a^{-1}(ca)^2a$$
$$\cdot c(ca)^2c^{-1}c^{-1}(ac)^2ca^{-1}c^{-1}(ac)^2cac^{-1}a^{-1}(ca)^2ac = 1 \ .$$

Replace $b^{-1}a^{-1}b^{-1}$ using Lemma 9 (2) to obtain

$$ba\ b^{-1}c^{-1}a^2b^{-1} \cdot c^{-1}ac^{-1}acba^{-1}cac^{-1}a$$
$$\cdot (ac)^2(ca)^2a^{-1}(ca)^2ac^{-1}(ac)^2cc(ca)^2c^{-1}c^{-1}a^{-1}(ca)^2ac = 1 \ .$$

By Lemma 9 (2), Lemma 5 we have

$$c^{-1}a^2b^{-1}c^{-1}b^{-1}a^{-1}c^{-1}bc^{-1}ac^{-1}acba^{-1}cac^{-1}a \cdot (ca)^2a^{-1}(ca)^2ac^{-1}(ac)^2cc(ca)^2c^{-1} = 1 \ .$$

By Lemma 9 (3) and Lemma 5 we have

$$c^{-1}a^2ca^{-1}c^2b^{-1}c^{-1} \cdot \left(c^{-1}a\right)^2 \cdot cba^{-1}cac^{-1}a \cdot (ca)^2$$
$$\cdot a(ac)^2a^{-1}a^{-1}(ca)^2ac^{-1}(ac)^2cc(ca)^2c^{-1}c^{-1}a^{-1}(ca)^2ac \ .$$

By Lemmas 5 and 6 we have

$$\left(c^{-1}a\right)^4 = (ac)^2(ca)^2a(ac)^2a^{-1}a^{-1}(ca)^2ac(ca)^2c^{-1}a^{-1}c^{-1}(ac)^2ca \ .$$

Conjugate the relation Lemma 9 (12) with a^2 and compare the result with Lemma 9 (12), then we obtain

$$(ac)^2c(ca)^2c^{-1}a^{-1}(ca)^2aa^{-1}c^{-1}(ac)^2ca = 1 \ .$$

Using this relation in the above yields the first part of (3). Applying Lemma 5 yields the second part.

(4) Apply the automorphism induced by p on $\langle a, b, c\rangle$ to the formulae of (3) to obtain (4).

(5) Apply to (3) the automorphism induced by p^2 on $\langle a, b, c\rangle$ and compare the result with the original formula

$$\text{(I)} \qquad \begin{cases} c(ca)^2c^{-1} = (ac)^2(ca)^2a(ac)^2a^{-1} \ , \\ c^{-1}(ac)^2c = (ac)^2(ca)^2a^{-1}(ca)^2a \ , \\ a^{-1}c^{-1}(ac)^2ca = (ca)^2a(ac)^2a^{-1}a^{-1}(ca)^2a \ , \\ c^{-1}a^{-1}(ca)^2ac = (ac)^2a(ac)^2a^{-1}a^{-1}(ca)^2a \ . \end{cases}$$

Conjugate with b, a^2, b^{-1} and use Lemma 5 to obtain the second, third and fourth formula by the first formula.

Apply to (I) the automorphism induced by p to obtain

$$\text{(II)}\quad \begin{cases} a^{-1}(b^{-1}abc)^2a = (abcb^{-1})^2\cdot(b^{-1}abc)^2\cdot a^{-1}(cb^{-1}ab)^2a\ , \\ a(abcb^{-1})^2a^{-1} = (abcb^{-1})^2\cdot(b^{-1}abc)^2\cdot(cb^{-1}ab)^2\ , \\ a^2(cb^{-1}ab)^2a^2 = (b^{-1}abc)^2(cb^{-1}ab)^2a^{-1}(cb^{-1}ab)^2a\ , \\ a(cb^{-1}ab)^2a^{-1} = (abcb^{-1})^2(cb^{-1}ab)^2a^{-1}(cb^{-1}ab)^2a\ . \end{cases}$$

Using Lemma 10 (2) and Lemma 1 we have

$$qa^{-1}(ca)^2aq^{-1} = aD^{-1}d(ac)^2d^{-1}Da^{-1}\ .$$

By Lemma 2 (36), (42) follows

$$\begin{aligned} qa^{-1}(ca)^2aq^{-1} &= aD^{-1}(Dbabca^{-1}Fbc^{-1}a^{-1}b^{-1}Fac^{-1}b^{-1}a^{-1}b^{-1}D^{-1})^2Da^{-1} \\ &= ababca^{-1}F(ca)^2Fac^{-1}b^{-1}a^{-1}b^{-1}a^{-1}\ . \end{aligned}$$

Use Lemma 10 (3), (4) and (I) and (II) in Lemma 9 (22), then follows

$$FcaF = (ac)^2(ca)^2a(ac)^2a^{-1}a^{-1}(ca)^2a(babc)^2abcab^{-1}a^{-1}\ .$$

Insert this in the above formula to obtain, using Lemma 5,

$$qa^{-1}(ca)^2aq^{-1} = ababca^{-1}\cdot((babc)^2abcab^{-1}a^{-1})^2\cdot ac^{-1}b^{-1}a^{-1}b^{-1}a^{-1}\ .$$

Apply Lemma 8, then follows

$$\begin{aligned} qa^{-1}(ca)^2aq^{-1} = {} & ababca^{-1}\cdot a^{-1}(b^{-1}abc)^2a\cdot(abcb^{-1})^2\cdot a(cb^{-1}ab)^2a^{-1}a(abcb^{-1})^2a^{-1}(cb^{-1}ab)^2 \\ & \cdot a^2(cb^{-1}ab)^2a^2ab(ca)^2b^{-1}a^{-1}\cdot ac^{-1}b^{-1}a^{-1}b^{-1}a^{-1} \end{aligned}$$

Lemma 5 and (II) imply

$$qa^{-1}(ca)^2aq^{-1} = ababca^{-1}a^{-1}(b^{-1}abc)^2a(abcb^{-1})^2(ca)^2ac^{-1}b^{-1}a^{-1}b^{-1}a^{-1}\ .$$

Lemma 9 (12), Lemma 5 and (I) imply

$$qa^{-1}(ca)^2aq^{-1} = (ca)^2\ .$$

A comparison with Lemma 10 (1) shows

$$a^{-1}(ca)^2a = (ca)^2\ .$$

Applying the automorphism induced by p^2 yields

$$c^{-1}(ac)^2c = (ac)^2\ .$$

Then the statement on $\langle (ac)^2(ca)^2\rangle$ follows by Lemma 5. To obtain the statement on $\langle (abcb^{-1})^2(b^{-1}abc)^2\rangle$ conjugate with p . One obtains the relations

$$a^{-1}(cb^{-1}ab)^2a = (b^{-1}abc)^2\ ,$$

$$(cb^{-1}ab)^2 = (abcb^{-1})^2\ .$$

Use Lemmas 6 and 8 and Lemma 10 (3), (4) to conclude the remaining statements.

LEMMA 11. *In* $\langle a, b, c\rangle$ *we have the following relations*

(1) $(b^{-1}cbc)^2 = (c^{-1}a)^2\cdot(babc)^2$;

(2) $(a^{-1}bab)^2 = (ac^{-1})^2\cdot(abcb)^2$;

(3) $(bcab^{-1}ca)^2 = (ac^{-1})^2\cdot(c^{-1}a)^2$;

(4) $cabacb^{-1}a^{-1}c^{-1} = bacb^{-1}$

(5) $(b^{-1}a^2ba^2)^2 = 1$;

(6) $(b^2ca)^2 = (ac^{-1})^2$;

(7) $(cb^{-1}a^{-1}b)^2 = (babc)^2$;

(8) $bcbc^{-1}a^{-1}b^{-1}c^{-1}b^{-1}a^{-1}c^{-1} = (c^{-1}a)^2$;

(9) $(bab^{-1}c^{-1})^2 = (ac^{-1})^2\cdot(c^{-1}a)^2(babc)^2$;

(10) $b^{-1}a^{-1}b^{-1}\cdot ac\cdot bab\cdot a^{-1}c^{-1} = (ca)^2(c^{-1}a)^2(abcb)^2\cdot(babc)^2$;

(11) $b^{-1}a^{-1}b^{-1}\cdot ca\cdot bab\cdot c^{-1}a^{-1} = (c^{-1}a)^2(ac^{-1})^2(abcb)^2\cdot(babc)^2$;

(12) $b^2acb^2c^{-1}a^{-1} = (ac)^2(ac^{-1})^2(abcb)^2\cdot(babc)^2$;

(13) $(abcb)^2 = (babc)^2$, $(c^{-1}a)^2 = (ac^{-1})^2$;

(14) $(abcab)^4 = 1$;

(15) $b^{-1}cabca^{-1}b^{-1}cabca^{-1} = 1$;

(16) $b^{-1}caba^2b^{-1}caba^2 = (c^{-1}a)^2$;

(17) $(a^{-1}b^{-1}ab)^2 = (ca)^2\cdot(abcb^{-1})^2$, $(b^{-1}c^{-1}bc)^2 = (b^{-1}abc)^2\cdot(ca)^2$;

(18) $(ab)^4 = (ba)^4$;

(19) $(ab)^4 \in Z(\langle a, b, c\rangle)$;

(20) $ba^2c^2b^{-1}c^2a^2 = (ac)^2(babc)^2\cdot(ab)^4$;

(21) $ba^2b^{-1}acba^2b^{-1}ac = (c^{-1}a)^2$;

(22) $(ba)^4 = (cb)^4$, $(ab)^4 = (bc)^4$;

(23) $b^2cb^{-1}ac^{-1}b^{-1}a^{-1} = (ac^{-1})^2(ac)^2(ca)^2\cdot(ab)^4$;

(24) $ba^{-1}c^{-1}b^{-1}c^{-1}a^{-1} = (c^{-1}a)^2(babc)^2\cdot(abcb^{-1})^2\cdot(b^{-1}abc)^2\cdot(ab)^4$;

(25) $(ab)^2 = (ca)^2\cdot(abcb^{-1})^2\cdot(b^{-1}abc)^2\cdot(abcb)^2\cdot(ab)^4\cdot(ba)^2$,

$(bc)^2 = (ac)^2(ca)^2(b^{-1}abc)^2(ac^{-1})^2\cdot(ab)^4\cdot(cb)^2$;

(26) $(ab)^4 = (abcb^{-1})^2\cdot(b^{-1}abc)^2\cdot(babc)^2\cdot(c^{-1}a)^2$;

(27) $(ac)^2(ca)^2 = (abcb^{-1})^2(b^{-1}abc)^2$;

(28) $(ac)^2 = (ca)^2$, $(abcb^{-1})^2 = (b^{-1}abc)^2$.

Proof. (1) By Lemma 9 (2), Lemma 10 (5) we have

$$b^{-1}cbc = (ca)^2(a^{-1}c)^2aba^{-1}b^{-1}a^2 .$$

Then (1) follows by Lemma 9 (14).

(2) We use the automorphism induced by p^{-1} in Lemma 11 (1). Then Lemma 11 (2) follows.

(3) By Lemma 2 (2) and Lemma 4 (1),

$$bcab^{-1}ca\cdot(ca)^2(ca^{-1})^2\cdot bca\cdot(ca)^2(ca^{-1})^2b^{-1}ca = 1 .$$

Hence Lemma 10 (5) implies Lemma 11 (3).

(4) By Lemma 2 (25) we have

$$bcFac^{-1}b^{-1}FDbac^2b^{-1}a^{-1}c^{-1} = c^{-1}b^{-1}cabc^2a^{-1}b^{-1}D^{-1}Fbca^{-1}F .$$

Insert this in Lemma 2 (28); then

$$Fac^{-1}b^{-1}FDbac^2b^{-1}Fac^{-1}b^{-1}FDba = c^{-1}b^{-1}cac^{-1}b^{-1}cabc^2 .$$

A comparison with Lemma 2 (26) yields

$$c^2a^{-1}b^{-1}cabc^2a = bacb^{-1} .$$

Then Lemma 11 (4) follows by Lemma 9 (3).

(5) By Lemma 9 (1), (2), Lemma 10 (5)

$$bab = ca^2bc^{-1}bac^2 = cba^2cba\cdot(ac)^2(ca)^2 .$$

Using Lemma 10 (5) we obtain

$$b^{-1}a^2ba^2 = (ca)^2(ca^{-1})^2c^{-1}\cdot bc^2b^{-1}c^2\cdot c .$$

Then Lemma 9 (5) implies Lemma 11 (5).

(6) follows immediately by Lemma 4 (1) and Lemma 10 (5).

(7) Applying the automorphism induced by p yields Lemma 11 (7).

(8) follows immediately by Lemma 9 (3) and Lemma 10 (5).

(9) By Lemma 10 (5),

$$bab^{-1}c^{-1}\cdot a^{-1}\cdot ba^{-1}b^{-1}a^{-1}\cdot aca = (ca)^2bab^{-1}\cdot ac\cdot ba^{-1}b^{-1}ca \ .$$

Replace $ba^{-1}b^{-1}a^{-1}$ using Lemma 9 (14) and bab^{-1} on the right-hand side using Lemma 9 (2), then

$$\left(a^{-1}c\right)^2(babc)^2\left(bab^{-1}c^{-1}\right)^2 = c^{-1}a^2c\cdot\left(c^{-1}b^{-1}\right)^2\cdot ca\cdot(bc)^2\cdot c^{-1}a^2c^2a \ .$$

Then Lemma 9 (3) and Lemma 10 (5) imply Lemma 11 (9).

(10) By Lemma 4 (1) we have

$$ababac^{-1}a^2b^{-1}a^{-1}b^{-1}a^2c^{-1} = abab^{-1}\cdot a^{-1}c^{-1}\cdot ba^{-1}b^{-1}a^{-1}\cdot a^{-1}c^{-1} \ .$$

Replacing $abab^{-1}$ using Lemma 9 (14) yields, by Lemma 10 (5),

$$ababac^{-1}a^2b^{-1}a^{-1}b^{-1}a^2c^{-1} = \left(ac^{-1}\right)^2ba^{-1}b^{-1}\cdot ac\cdot bab^{-1}\cdot a^{-1}c^{-1} \ .$$

Replacing bab^{-1} using Lemma 9 (2) yields, by Lemma 10 (5),

$$ababac^{-1}a^2b^{-1}a^{-1}b^{-1}a^2c^{-1} = a^{-1}b^{-1}cba^{-1}\cdot c^{-1}b^{-1}c^{-1}b\cdot c^{-1} \ .$$

Replacing $c^{-1}b^{-1}c^{-1}b$ using Lemma 11 (1) yields, by Lemma 10 (5),

$$ababac^{-1}a^2b^{-1}a^{-1}b^{-1}a^2c^{-1} = \left(ac^{-1}\right)^2(abcb)^2\left(a^{-1}b^{-1}cb\right)^2 \ .$$

Then Lemma 11 (9) and Lemma 10 (5) imply Lemma 11 (10).

(11) Applying to Lemma 11 (10) the automorphism induced by p^2 yields Lemma 11 (11) by Lemma 10 (5).

(12) By Lemma 11 (8) and Lemma 4 (1)

$$b^2acb^2c^{-1}a^{-1} = \left(ca^{-1}\right)^2\cdot\left(c^{-1}a\right)^2\cdot bc^{-1}b\cdot cab^{-1}cb^{-1}c^{-1}a^{-1} \ .$$

Replacing $bc^{-1}b$ using Lemma 9 (1) yields, by Lemma 10 (5),

$$b^2acb^2c^{-1}a^{-1} = a^2c^{-1}\cdot babacb^{-1}a^{-1}b^{-1}\cdot ca^2c^{-1}a^{-1} \ .$$

Use Lemma 11 (11) to conclude Lemma 11 (12).

(13) We have

$$abcb^{-1}c\cdot abc^{-1}b^{-1}\cdot a^2c^{-1} = ab^2\cdot b^{-1}cb^{-1}\cdot ca\cdot bc^{-1}bb^2a^2c^{-1} \ .$$

Replacing $abc^{-1}b^{-1}$ using Lemma 11 (7) and $bc^{-1}b$ using Lemma 9 (1) yields, by Lemma 10 (5),

$$(babc)^2abc\cdot b^{-1}cbc\cdot b^{-1}ac^{-1} = \left(c^{-1}a\right)^2\left(ac^{-1}\right)^2ab^2\cdot ac^2\cdot b^{-1}a^{-1}b^{-1}\cdot ac\cdot bab\cdot c^2a^{-1}ba^2c^{-1} \ .$$

Replacing $b^{-1}cbc$ using Lemma 11 (1) and applying Lemma 11 (10) to the right hand side yields

$$\left(c^{-1}a\right)^2\left(ac^{-1}\right)^2 = ab^2acb^2a^2c^{-1} .$$

A comparison with Lemma 11 (12) furnishes the first part of Lemma 11 (13). The second part is obtained by applying the automorphism induced by p .

(14) By Lemma 9 (1), (13)

$$ababc = (ac)^2\left(c^{-1}a\right)^2(babc)^2a^{-1}b^{-1}cb^{-1}a^{-1} .$$

Use Lemma 10 (5) to conclude

$$\begin{aligned} cab\cdot\left(c^{-1}b^{-1}a^{-1}b^{-1}a^{-1}\right)^4b^{-1}a^{-1}c^{-1} &= \left(ac^{-1}\right)^2cabc^{-1}b^{-1}a^{-1}c^{-1}bc^2\cdot c^{-1}a^{-1}b^{-1}a^{-1}c^{-1}bab^{-1}a^{-1}c^{-1} \\ &= \left(ac^{-1}\right)^2cabc^{-1}b^{-1}a^{-1}c^{-1}bc^2b^{-1}a^{-1}\cdot c^{-1}bc^{-1}b^{-1}\cdot a^{-1}c^{-1} \\ &\qquad \text{by Lemma 11 (4)} \\ &= (babc)^2c\cdot abc^{-1}b^{-1}\cdot a^{-1}c^{-1}bc^2\cdot b^{-1}a^{-1}bc\cdot b^{-1}ca^{-1}c^{-1} \\ &\qquad \text{by Lemma 11 (1)} \\ &= \left(a^{-1}c\right)^2(babc)^2cbcb^{-1}a^2c^{-1}\cdot bcb^{-1}a^{-1}\cdot a^{-1}c^2 \\ &\qquad \text{by Lemma 11 (7)} \\ &= cbcb\cdot b^2acb^2b^{-1}c^{-1}b^{-1}a^{-1}c^2 \quad \text{by Lemma 11 (7).} \end{aligned}$$

Then Lemma 11 (12), Lemma 9 (3), Lemma 10 (5) imply Lemma 11 (14).

(15) As in the proof of Lemma 11 (14)

$$\left(c^{-1}b^{-1}a^{-1}b^{-1}a^{-1}\right)^4 = \left(ac^{-1}\right)^2c^{-1}b^{-1}a^{-1}c^{-1}bca^{-1}b^{-1}a^{-1}c^{-1}ba .$$

A comparison with Lemma 11 (14) yields Lemma 11 (15).

(16) By $(abc)^4 = 1$ and Lemma 4 (1)

$$b^{-1}cabcab^{-1}a^{-1}c^{-1}ba^{-1}c^{-1} = 1 .$$

A comparison with Lemma 11 (15) yields Lemma 11 (16).

(17) Use Lemma 11 (10) to conclude

$$b^2\cdot babca\cdot a^{-1}b^{-1}abc = \left(b^{-1}abc\right)^2 ,$$

$$b^2c^{-1}a^{-1}b^2\cdot b^{-1}aba^{-1}b^{-1}abc = \left(c^{-1}a\right)^2\left(b^{-1}abc\right)^2 .$$

Then we obtain (17) by Lemma 11 (12) and Lemma 10 (5). Transforming by p yields the second part.

(18) From (viii) follows $(ba)^8 = 1$. Then use Lemma 9 (14) to conclude

$$\begin{aligned}(ab)^4(ba)^4 &= abab\cdot abab^{-1}\cdot b^{-1}aba\cdot baba \\ &= abab\cdot ba^{-1}b^{-1}a\cdot ab^{-1}a^{-1}b^2aba \\ &= (ac)^2\left(b^{-1}abc\right)^2a\cdot baba^{-1}\cdot bab^{-1}ab^{-1}a^{-1}b^2aba \quad \text{by Lemma 11 (17)} \\ &= (ac)^2\left(b^{-1}abc\right)^2\left(ac^{-1}\right)^2(abcb)^2a^2b^{-1}\cdot b^{-1}ab^{-1}a^{-1}\cdot b^2aba \quad \text{by Lemma 11 (2)} \\ &= (ac)^2\left(b^{-1}abc\right)^2a^{-1}\left(a^{-1}b^{-1}ab\right)^2a \quad \text{by Lemma 11 (2);}\end{aligned}$$

hence Lemma 11 (18) by Lemma 11 (17).

(19) The relations $a(ab)^4a^{-1} = (ab)^4$ and $b(ab)^4b^{-1} = (ab)^4$ follow by Lemma 11 (18), $c(ab)^4c^{-1} = (ab)^4$ follows by Lemma 11 (10).

(20) By Lemma 9 (1), (2)

$$(ab)^2 = aca^2bc^{-1}bac^2 ,$$

$$(ab)^2 = (ac)^2(ca)^2acba^2cba .$$

Hence

$$(ab)^4 = (ca)^2\left(c^{-1}a\right)^2acba^2c\cdot bcbc^{-1}\cdot bac^2 .$$

Then Lemma 11 (20) follows by Lemma 9 (13) and Lemma 10 (5).

(21) By Lemma 11 (11), (12)

$$b^{-1}cab = ab^{-1}acba^{-1}\cdot(ac)^2\left(ac^{-1}\right)^2 .$$

By Lemma 11 (10) and Lemma 4 (1)

$$b^{-1}cab = a^{-1}b^{-1}acba .$$

A comparison yields Lemma 11 (21).

(22) Use Lemma 9 (2), Lemma 10 (5) to conclude

$$(ba)^4 = bab\cdot a\cdot bab\cdot a ,$$

$$(ba)^4 = cbcb\cdot b^{-1}a^2bca\cdot aba^2cba^2 .$$

By Lemma 11 (16) and Lemma 4 (1)

$$\begin{aligned}(ba)^4 &= \left(ca^{-1}\right)^2(ca)^2\cdot(cb)^2\cdot cab^{-1}a^{-1}\cdot a^{-1}bab\cdot a^2cba^2 \\ &= (ca)^2(abcb)^2(cb)^2\cdot c\cdot ab^{-1}a^{-1}b^{-1}\cdot a^{-1}b^{-1}a^{-1}cba^2 \quad \text{by Lemma 11 (2)} \\ &= (ca)^2(cb)^3aba^2b^{-1}c^{-1}a^{-1}\cdot a^2ba^2 \quad \text{by Lemma 11 (2).}\end{aligned}$$

Then the first part of Lemma 11 (22) follows by Lemma 11 (21), (5). The second part follows by Lemma 11 (18), (19).

(23) Use Lemma 11 (1) to conclude

$$\begin{aligned} b^2cb^{-1}ac^{-1}b^{-1}a^{-1} &= (ac^{-1})^2b\cdot bcb^{-1}c\cdot a^{-1}b^{-1}a^{-1} \\ &= (ac)^2(abcb)^2ba\cdot a^{-1}c^{-1}bacb^{-1}\cdot c^{-1}a^2b^{-1}c^{-1}a^{-1} \quad \text{by Lemma 9 (2)} \\ &= (ac)^2(ca)^2(abcb)^2babc^{-1}a^{-1}b^{-1}a^{-1}b^{-1}\cdot bc^2a^2b^{-1}c^{-1}a^{-1} \end{aligned}$$

by Lemma 11 (4).

By Lemma 11 (13), (11)

$$b^2cb^{-1}ac^{-1}b^{-1}a^{-1} = (ac)^2(ca)^2(abcb)^2a^{-1}c^{-1}bc^2a^2b^{-1}c^{-1}a^{-1} ,$$

hence

$$b^2cb^{-1}ac^{-1}b^{-1}a^{-1} = (abcb)^2a^{-1}c^{-1}ba^2c^2b^{-1}c^{-1}a^{-1} .$$

Then Lemma 11 (23) follows by Lemma 11 (20).

(24) Apply the automorphism induced by p^{-1} to Lemma 11 (23) and use Lemma 10 (5), Lemma 11 (22) to conclude Lemma 11 (24).

(25) By Lemma 9 (2)

$$(ab)^2 = (ac)^2\cdot c^{-1}a^{-1}ba^{-1}c^{-1}b^{-1}\cdot(ba)^2\cdot(ca^{-1})^2(ac)^2(ca)^2 .$$

Then Lemma 11 (25) follows by Lemma 11 (24). The second part is obtained by transformation with p .

(26) By Lemma 3 (4), Lemma 4 (1) and Lemma 10 (5)

$$D^{-1}abD = (ca)^2\cdot bc^{-1}\cdot abc\cdot b^{-1} .$$

Replacing abc using Lemma 11 (23) yields

$$D^{-1}abD = (ac^{-1})^2(ac)^2(ab)^4\cdot bc^{-1}b^2ca\cdot a^{-1}b^{-1}ab^{-1} .$$

Use Lemma 4 (1), Lemma 11 (2), Lemma 10 (5) to conclude

$$D^{-1}abD = (ac^{-1})^2(abcb)^2(ac)^2(ca)^2(ab)^4\cdot ba\cdot b^{-1}a^{-1}ba .$$

By Lemma 11 (17)

$$D^{-1}abD = (ac^{-1})^2(abcb)^2(abcb^{-1})^2(ac)^2(ab)^4\cdot ab .$$

This computation can also be carried through in the following way:

$$D^{-1}abD = (ca)^2bc^{-1}a\cdot bcbc\cdot c^{-1}ba^{-1}\cdot ab .$$

By Lemma 11 (25)

$$D^{-1}abD = (ab)^4\cdot(ac)^2(abcb^{-1})^2(ac^{-1})^2bc^{-1}ac\cdot bcbc^{-1}\cdot ba^{-1}\cdot ab .$$

By Lemma 9 (13)

$$D^{-1}abD = (ab)^4\cdot(ac)^2\cdot(babc)^2(abcb^{-1})^2\cdot(c^{-1}a)^2\cdot ba^{-1}c^{-1}b^{-1}c^{-1}a^{-1}\cdot ab .$$

By Lemma 11 (24)

$$D^{-1}abD = (ac)^2(b^{-1}abc)^2 \cdot ab \ .$$

Then Lemma 11 (26) follows by comparing the formulae for $D^{-1}abD$.

(27) Apply to Lemma 11 (26) the automorphism induced by p and use Lemma 11 (22) to deduce Lemma 11 (27).

(28) By $(abc)^4 = 1$, Lemma 4 (1), Lemma 11 (24), (26), (27) we have

$$abcab^{-1} \cdot b^2cabcabc = 1 \ ,$$

$$(c^{-1}a)^2(ac)^2 \cdot ac^{-1}aca^{-1} \cdot bcab^{-1}c = 1 \ .$$

Use Lemma 11 (24), (27), (26) to conclude the first part of Lemma 11 (28). The second part is obtained by transformation with p .

Using Lemma 10 (5), the relations Lemma 11 (13), (25), (28) show that

$$\langle (ac)^2,\ (abcb^{-1})^2,\ (ac^{-1})^2,\ (abcb)^2,\ (ab)^4 \rangle$$

is an elementary abelian normal subgroup of exponent 2 in the centre of $\langle a, b, c \rangle$. It is of order 2^4 at most, and is generated by $(ac)^2$, $(abcb^{-1})^2$, $(ac^{-1})^2$, $(abcb)^2$. In the following we shall use the simplified relations obtained by Lemma 11 (26), (28) without any special indication.

LEMMA 12. *We have the transformation formulae:*

(1) $D^{-1}a^2D = (ca)^2 \cdot (ac^{-1})^2 \cdot (abcb^{-1})^2 \cdot (abcb)^2 \cdot a^2$;

(2) $D^{-1}b^2D = (ca)^2 \cdot (ac^{-1})^2(abcb^{-1})^2 \cdot (abcb)^2 \cdot b^2$;

(3) $D^{-1}c^2D = (ca)^2(ac^{-1})^2(abcb^{-1})^2 \cdot (abcb)^2 \cdot c^2$;

(4) $D^{-1}abD = (ac)^2 \cdot (b^{-1}abc)^2 \cdot ab$;

(5) $D^{-1}acD = ac$;

(6) $D^{-1}bcD = (c^{-1}a)^2(abcb)^2 \cdot bc$;

(7) $D^2 = (ca)^2 \cdot (abcb^{-1})^2$;

(8) $Fa^2F = a^2$;

(9) $Fca^{-1}F = (babc)^2 \cdot (c^{-1}a)^2 \cdot ca^{-1}$;

(10) $FacF = (abcb)^2 \cdot (c^{-1}a)^2 \cdot ac$;

(11) $Fbcb^{-1}a^{-1}F = (abcb)^2 \cdot (c^{-1}a)^2 \cdot bcb^{-1}a^{-1}$;

(12) $Fabcb^{-1}F = (abcb)^2(c^{-1}a)^2abcb^{-1}$;

(13) $Fb^2F = b^2$;

(14) $Fab^2a^{-1}F = ab^2a^{-1}$;

(15) $Fabab^{-1}F = abab^{-1}$;

(16) $Fbab^{-1}a^{-1}F = bab^{-1}a^{-1}$;

(17) $D^{-1}aD = ab^{-1}a^{-1}FabF$;

(18) $D^{-1}bD = (abcb)^2\cdot(c^{-1}a)^2\cdot a^{-1}FabF$;

(19) $D^{-1}cD = (abcb)^2\cdot(c^{-1}a)^2\cdot(ac)^2(abcb^{-1})^2\cdot cb^{-1}a^{-1}FabF$;

(20) $D^{-1}FD = F$.

Proof. (1) By Lemma 3 (1), Lemma 4 (1), Lemma 10 (5) follows immediately

$$D^{-1}a^2D = (ca)^2(ac^{-1})^2bc^{-1}b^{-1}a^{-1}\cdot a^{-1}bcb^{-1} .$$

Use Lemma 11 (7) to conclude Lemma 12 (1).

(2) By Lemma 3 (2), Lemma 4 (1), Lemma 10 (5)

$$D^{-1}b^2D = (ca)^2(ac^{-1})^2\cdot bc^{-1}b^{-1}a^{-1}b^2\cdot bc^{-1}b^{-1}a^{-1}\cdot(abcb^{-1})^2 .$$

Hence Lemma 12 (2).

(3) By Lemma 3 (3), Lemma 4 (1), Lemma 10 (5)

$$D^{-1}c^2D = (ca)^2(c^{-1}a)^2\cdot bc^{-1}b^{-1}a^{-1}c^2ac^{-1}\cdot cbcb^{-1} .$$

Lemma 11 (1), Lemma 10 (5) imply

$$D^{-1}c^2D = (c^{-1}a)^2(babc)^2(ca)^2\cdot(ac)^2\cdot(bc^{-1}b^{-1}c)^2\cdot c^2 .$$

Use Lemma 11 (17) to conclude Lemma 12 (3).

(4) See the proof of Lemma 11 (26).

(5) By Lemma 3 (6), Lemma 4 (1), Lemma 10 (5)

$$D^{-1}acD = bc^{-1}b^{-1}cabcb^{-1} .$$

Use Lemma 4 (1) and Lemma 11 (24) to conclude Lemma 12 (5).

(6) By Lemma 3 (8), Lemma 4 (1)

$$D^{-1}bcD = bc^{-1}b^{-1}a^{-1}bcab^{-1}\cdot b^2\cdot cb^{-1}c^{-1}b^{-1}\cdot bc\cdot(ac^{-1})^2 .$$

By Lemma 9 (13), Lemma 11 (24)

$$D^{-1}bcD = b\cdot c^{-1}b^{-1}cb\cdot b^{2}cbc^{-1}\cdot bc\cdot (abcb)^{2}\cdot (ac)^{2} \ .$$

By Lemma 11 (17) follows

$$D^{-1}bcD = (abcb)^{2}\cdot \left(abcb^{-1}\right)^{2}\cdot c^{-1}\cdot bcbc\cdot c^{-1}\cdot bcbc^{-1}\cdot bc \ .$$

Use Lemma 11 (25), Lemma 9 (13) to conclude Lemma 12 (6).

(7) follows by Lemma 3 (10) and Lemma 4 (1).

(8) is Lemma 2 (19).

(9) follows by Lemma 9 (22), Lemma 11 (24), Lemma 12 (8).

(10) follows by Lemma 9 (18), Lemma 11 (28), (24).

(11) By Lemma 9 (15), Lemma 12 (8) follows

$$Fbcb^{-1}a^{-1}F = b\cdot b^{2}cab^{2}\cdot b\cdot baba \ .$$

By Lemma 4 (1), Lemma 11 (25), (27), (28) follows

$$Fbcb^{-1}a^{-1}F = (ab)^{4}bcb^{-1}a^{-1} \ .$$

Hence Lemma 12 (11) follows by Lemma 11 (28).

(12) By Lemma 9 (19), Lemma 10 (5) follows

$$Fabcb^{-1}F = (ac)^{2}a^{-1}b^{-1}a^{-1}\cdot cab^{2}\cdot c^{2}b^{-1}$$

implying

$$Fabcb^{-1}F = \left(ac^{-1}\right)^{2}(ac)^{2}a^{-1}b^{-1}a^{-1}b^{-1}\cdot b^{-1}a^{-1}cb^{-1}$$

by Lemma 4 (1). Then Lemma 12 (12) follows by Lemma 11 (25), (28).

(13) By Lemma 2 (22), (23) and $(abc)^{4} = 1$ follows

$$Fb^{2}ca^{-1}F = (ac)^{2}\cdot bcb^{-1}cabb^{2}cb^{-1}$$

implying

$$\begin{aligned} Fb^{2}ca^{-1}F &= \left(ac^{-1}\right)^{2}b^{2}\cdot b^{-1}a^{-1}b^{-1}a^{-1}\cdot ab^{-1}cb^{-1} \quad \text{by Lemma 11 (24)} \\ &= (ca)^{2}\cdot b^{2}a^{-1}b^{-1}a^{-1}b^{-1}ab^{-1}\cdot cb^{-1} \quad \text{by Lemma 11 (25).} \end{aligned}$$

Then Lemma 12 (13) follows by Lemma 11 (2), (24) and Lemma 12 (9).

(14) By Lemma 9 (20) and Lemma 12 (10) follows

$$\begin{aligned} Fab^{2}a^{-1}F &= \left(c^{-1}a\right)^{2}\cdot a^{-1}b^{-1}a^{-1}\cdot a^{-1}c^{-1}b^{2}c^{2}b^{-1}ac^{-1}b^{-1}a^{-1}b^{-1}a^{-1} \\ &= (ca)^{2}a^{-1}b^{-1}a^{-1}b^{-1}\cdot b^{-1}a^{-1}cb^{-1}ac^{-1}b^{-1}a^{-1}b^{-1}a^{-1} \quad \text{by Lemma 4 (1)} \\ &= (abcb)^{2}\left(abcb^{-1}\right)^{2}\cdot b\cdot c^{-1}b^{-1}c^{-1}b^{-1}\cdot a^{-1}b^{-1}a^{-1} \quad \text{by Lemma 11 (25), (26).} \end{aligned}$$

Hence Lemma 12 (14) follows by Lemma 11 (25), (24).

(15) By Lemma 12 (14) and Lemma 2 (22)

$$Fabab^{-1}F = ab\cdot ba^2c^2b^{-1}a^{-1}c^2b^{-1} .$$

Then Lemma 12 (15) follows by Lemma 11 (20) and Lemma 10 (5).

(16) By Lemma 2 (22), (20)

$$FbabccaF = bc^2a^{-1}ba^{-1} .$$

By Lemma 12 (8), (9), (14)

$$Fbab^{-1}a^{-1}F = bc^2a^{-1}ba^2c^2b^{-1}b^{-1}a^{-1} .$$

Lemma 11 (20) and Lemma 10 (5) imply Lemma 12 (16).

(17) By Lemma 2 (32), Lemma 12 (8), (7), (9)

$$\begin{aligned} D^{-1}aD &= a^{-1}b^{-1}aFac^{-1}abacc^2F \\ &= a^{-1}b^{-1}aFc^2\cdot abcb^{-1}\cdot bcF \quad \text{by Lemma 11 (24), Lemma 12} \\ &= \left(abcb^{-1}\right)^2\cdot a^{-1}b^{-1}aFc^2bc^{-1}\cdot b^{-1}a^{-1}bc\cdot F \quad \text{by Lemma 12} \\ &= \left(abcb^{-1}\right)^2(abcb)^2a^{-1}b^{-1}ac^2bc^2b^{-1}\cdot FabF \quad \text{by Lemma 2 (20), Lemma 12.} \end{aligned}$$

Then Lemma 12 (17) follows by Lemma 11 (24), (1), (17).

(18) follows by Lemma 12 (18), (2), (4).

(19) follows by Lemma 12 (18), (6), (3).

(20) By Lemma 2 (34), Lemma 12

$$D^{-1}FD = \left(ac^{-1}\right)^2\cdot(abcb)^2\cdot\left(cbcb^{-1}\right)^2F .$$

Then Lemma 12 (20) follows by Lemma 11 (1).

We shall now describe the normal subgroup which is generated by the fourth powers in G .

DEFINITION.

$$v_1 = b^{-1}a^{-1}Dba = \left(p^2q\right)^4 ,$$

$$v_2 = ab^{-1}a^{-1}Db ,$$

$$v_3 = c^{-1}b^{-1}cb^{-1}D^{-1}cb^{-1}Fac^{-1}b^{-1}a^{-1}b^{-1}Fb ,$$

$$v_4 = (ac)^2 = \left(qpq^{-1}p\right)^4 ,$$

$$v_5 = \left(abcb^{-1}\right)^2 ,$$

$$v_6 = \left(a^{-1}c\right)^2 ,$$

$$v_7 = (abcb)^2 ,$$

$$v_8 = bc^{-1}b^{-1}cb^{-1}D^{-1}cb^{-1}Fac^{-1}b^{-1}a^{-1}b^{-1}F ;$$

$$V := \langle v_1, \ldots, v_8\rangle .$$

We shall show that V is a normal subgroup in G generated by fourth powers.

LEMMA 13. *We have the transformation formulae*

(1) $pv_1p^{-1} = v_3$;

(2) $pv_2p^{-1} = v_8$;

(3) $pv_3p^{-1} = v_4v_2$;

(4) $pv_4p^{-1} = v_5$;

(5) $pv_5p^{-1} = v_4$;

(6) $pv_6p^{-1} = v_7$;

(7) $pv_7p^{-1} = v_6$;

(8) $pv_8p^{-1} = v_4v_1$;

(9) $qv_1q^{-1} = v_4v_2$;

(10) $qv_2q^{-1} = v_6v_1$;

(11) $qv_3q^{-1} = v_7v_3$;

(12) $qv_4q^{-1} = v_4$;

(13) $qv_5q^{-1} = v_5$;

(14) $qv_6q^{-1} = v_7$;

(15) $qv_7q^{-1} = v_6$;

(16) $qv_8q^{-1} = v_5v_8$;

(17) V *is an elementary abelian normal subgroup of exponent* 2 *in* G ;

(18) $v_8 = v_1v_2v_3$;

(19) $v_7 = v_4v_5v_6$.

Proof. (1) follows immediately by Lemma 1 and Lemma 2 (47).

(2) follows immediately by Lemma 1 and Lemma 2 (47).

(3) By Lemmas 12, 1, and the definition of v_i

$$\begin{aligned} p^{-1}v_4\cdot v_2p &= p^{-1}\cdot\left(abcb^{-1}\right)^2aDb^{-1}a^{-1}b\cdot p \\ &= (ca)^2c^{-1}b^{-1}a^{-1}c^{-1}e^2cbca . \end{aligned}$$

If one computes p^4ep^{-4} using Lemma 1, then

$$ca \rightleftarrows e .$$

Hence Lemma 13 (3) follows by Lemma 11 (24), Lemma 2 (47), Lemma 12.

(4) to (7) follow by Lemma 1, Lemma 11 (28), Lemma 9 (13).

(8) Use the definition of v_i to conclude

$$pv_8p^{-1} = pb\cdot v_3b^{-1}\cdot p^{-1} .$$

Then Lemma 13 (8) follows by Lemma 1, Lemma 13 (3), Lemma 11 (24), Lemma 12.

(12) follows by Lemma 10 (1) and Lemma 11 (28).

(13) By Lemma 1

$$q\left(abcb^{-1}\right)^2q^{-1} = \left(deb^{-1}dbc^{-1}e^{-1}a^{-1}\right)^2 .$$

But

$$\begin{aligned} deb^{-1}dbc^{-1}ec &= q^2p^2\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot q^2\cdot pq^{-1}pq^{-1}\cdot pq \\ &= p^{-1}q^{-1}p^{-1}q^2\cdot pq^{-1}pq^{-1}\cdot p \\ &= q^{-1}\left(q^{-1}p^{-1}\right)^4q = 1 \quad \text{by (iv), (iii)} \end{aligned}$$

implying

$$q\left(abcb^{-1}\right)^2q^{-1} = c^{-1}e^{-2}a^{-1}c^{-1}e^{-2}a^{-1} .$$

Using $ca \rightleftarrows e$ as in the proof of Lemma 13 (3) yields

$$q\left(abcb^{-1}\right)^2q^{-1} = (ac)^2ae^4a^{-1} .$$

We apply the automorphism induced by p on Lemma 12 (7) obtaining

$$e^4 = (ca)^2\cdot\left(b^{-1}abc\right)^2 .$$

Hence the proof of Lemma 13 (13) is complete using Lemma 10 (5).

(14) By Lemma 1

$$\begin{aligned} q\left(ac^{-1}\right)^2q^{-1} &= dbcb^{-1}d^{-1}a^{-1}db\cdot cb^{-1}d^{-1}\cdot a^{-1} \\ &= dbcb^{-1}d^{-1}a^{-1}\cdot dbdb\cdot c^{-1}a^{-1} \quad \text{by Lemma 2 (49)} \\ &= dbcb^{-1}\cdot d^{-1}a^{-1}b^{-1}d^{-1}\cdot b^{-1}d^{-1}c^{-1}a^{-1} \quad \text{by Lemma 2 (49)} \\ &= dbcab^{-1}\cdot d^{-1}c^{-1}a^{-1} \quad \text{by Lemma 2 (49).} \end{aligned}$$

Then by Lemma 2 (39), (42), (36),

$$q\left(ac^{-1}\right)^2q^{-1} = cb^{-1}\cdot Fac^{-1}b^{-1}a^{-1}c^{-1}bca^{-1}Fbc^2a^{-1} .$$

Hence Lemma 13 (14) follows by Lemma 11 (24), Lemma 12, Lemma 11 (28).

(15) By Lemma 13 (6)

$$q(abcb)^2q^{-1} = qpq^{-1}p^{-1}\cdot pq\left(ac^{-1}\right)^2q^{-1}p^{-1}\cdot pqp^{-1}q^{-1}$$
$$= ap(abcb)^2p^{-1}\cdot a^{-1} .$$

Hence Lemma 13 (15) follows by Lemma 13 (7) and Lemma 10 (5).

(9) We have $v_1 = \left(p^2q\right)^2$; hence $p^2v_1p^{-2} = qv_1q^{-1}$. Then Lemma 13 (9) follows by Lemma 13 (3).

(10) By Lemma 1

$$qv_2q^{-1} = dbe^{-1}d^{-1}c^{-1}e^{-1}a^{-1}c^{-1}b^{-1}a^{-1} .$$

The definition of d, e and (v) implies immediately

$$(de)^2 = 1 .$$

This implies

$$qv_2q^{-1} = dbd^{-1}\cdot D\cdot ec^{-1}e^{-1}\cdot a^{-1}c^{-1}b^{-1}a^{-1} .$$

By Lemma 2 (39), (43),

$$qv_2q^{-1} = Dbc^{-1}a^{-1}c^{-1}b^{-1}a^{-1} .$$

Hence

$$qv_2q^{-1} = Dbc^{-1}a^{-1}c^{-1}\cdot b^{-1}a^{-1}b^{-1}a^{-1}\cdot Dba\cdot v_1^{-1}$$

implying

$$qv_2q^{-1} = D^2D^{-1}\cdot\left(c^{-1}a\right)^2\cdot a^{-1}b^{-1}Dba\cdot v_1^{-1}$$

by Lemma 11 (25). Then Lemma 13 (10) follows by Lemma 12, Lemma 10 (5).

(11) By Lemma 13 (1), (9), (8)

$$\begin{aligned}
qv_3q^{-1} &= v_5\cdot apv_2p^{-1}a^{-1}\\
&= v_5\cdot pc^{-1}b^2a^{-1}Dbabcp^{-1}\\
&= v_5p\left(c^{-1}a\right)^2\cdot(abcb)^2(ac)^2\left(b^{-1}abc\right)^2\cdot c^{-1}b^2\cdot a^{-1}bab\ cb^{-1}a^{-1}bap^{-1}\cdot v_3^{-1}\\
&\qquad\text{by Lemma 12, Lemma 13 (1)}\\
&= v_5\cdot p\left(ac^{-1}\right)^2\cdot(ca)^2c^{-1}b\cdot a^{-1}b^{-1}a^{-1}b^{-1}\cdot c^{-1}bp^{-1}\cdot v_3^{-1}
\end{aligned}$$

by Lemma 11 (2), Lemma 11 (24), Lemma 4 (1).

Hence Lemma 13 (11) follows by Lemma 11 (25), Lemma 11 (24) and Lemma 13 (5), (6).

(16) By Lemma 13 (2), (10), Lemma 1 and $v_1^2 = 1$

$$qv_8q^{-1} = v_7\cdot pc^{-1}b^{-1}a^{-1}a^{-1}b^{-1}D^{-1}ababcp^{-1}$$

$$= v_7\cdot p(ac)^2\cdot\left(c^{-1}a\right)^2\cdot c^{-1}b^{-1}a^{-1}baD^{-1}cab^{-1}a^{-1}Dbp^{-1}v_8^{-1}$$

by Lemma 12 (4), Lemma 11 (25).

Then Lemma 13 (12) follows by Lemma 12 (1), (3), (5), (4) and Lemma 13 (4), (6).

(17) Use the definition of v_i and the transformation formulae to conclude

$$v_1^2 = v_2^2 = \ldots = v_8^2 = 1 .$$

Lemma 10 (5), Lemma 11 (28), Lemma 12 show

$$\langle v_4, v_5, v_6, v_7 \rangle \leq Z(V) .$$

$\left(v_1v_2\right)^2 = \left(v_1v_3\right)^2 = \left(v_1v_8\right)^2 = 1$ follows by $v_1 = \left(p^2q\right)^4$ and Lemma 13 (1)-(16).

The other commutator formulae are obtained by transforming the above formulae by p and q .

(18) By the definition of v_1, v_3 and Lemma 12

$$v_1\cdot v_3 = (abcb)^2\cdot b^{-1}a^{-1}bac^{-1}b^{-1}cb^{-1}cb^{-1}\cdot abab\cdot caFa^{-1}Fb$$

$\Rightarrow$ $$v_1v_3 = (abcb)^2(ac)^2\cdot b^{-1}a^{-1}ba\cdot c^{-1}b^{-1}a^{-1}c^{-1}Fa^{-1}Fb$$ by Lemma 11 (25), (24)

$\Rightarrow$ $$v_1v_3 = \left(abcb^{-1}\right)^2\cdot\left(c^{-1}a\right)^2\cdot(ca)^2\cdot a^{-1}cb^{-1}c^{-1}a^{-1}\cdot a^{-1}Fa^{-1}Fb$$ by Lemma 11 (17), (7)

$\Rightarrow$ $$v_1v_3 = \left(ac^{-1}\right)^2\cdot\left(abcb^{-1}\right)^2\cdot(abcb)^2\cdot c^2ababFa\cdot\left(a^{-1}FabF\right)^2\cdot Fb^{-1}a^{-1}b$$

by Lemma 11 (24), (28).

Raise the formula Lemma 12 (18) to the second power and compare with Lemma 12 (2) to obtain

$$\left(a^{-1}FabF\right)^2 = (ac)^2\left(ac^{-1}\right)^2\cdot\left(abcb^{-1}\right)^2\cdot(abcb)^2\cdot b^2 .$$

We insert this in the above formulae obtaining, by Lemma 12,

$$v_1\cdot v_3 = (ac)^2\cdot c^2abababa^{-1}bFa^{-1}Fa$$

$\Rightarrow$ $$v_1v_3 = (ca)^2\cdot(abcb)^2\cdot\left(c^{-1}a\right)^2\cdot c^2b^{-1}a^2b\cdot Fa^{-1}Fa$$ by Lemma 11 (28).

By the definition of v_2, v_8 and Lemma 12

$$v_2v_8v_1 \cdot v_3 = (ca)^2 \cdot ab^{-1}a^{-1}b^2c^{-1}b^{-1}cb^{-1}cb^{-1}a \cdot c^{-1}b^{-1}a^{-1}b^{-1} \cdot c^2b^{-1}a^2b$$

$$= (abcb)^2 \cdot (c^{-1}a)^2 \cdot ab^{-1}a^{-1}b^2c^{-1}b^{-1}c \cdot b^{-1}cab \cdot ac^{-1}b^{-1}a^2b \quad \text{by Lemma 11 (25)}$$

$$= (abcb)^2 \cdot (ca)^2 \cdot ab^{-1}a^{-1}b^{-1} \cdot b^{-1}c^{-1}b^{-1}c^{-1}b^{-1}a^2b$$

by Lemma 11 (24), Lemma 4 (1)

$$= (abcb)^2 \cdot (ca)^2ab^{-1}a^{-1}c^{-1}b^{-1}c^{-1}ba^2b \quad \text{by Lemma 11 (22), (26), (2)}$$

$$= (babc)^2 \cdot (ca)^2 \cdot ab \cdot babaab \quad \text{by Lemma 11 (24).}$$

Then Lemma 13 (18) follows by Lemma 11 (25), (28).

(19) Lemma 13 (19) is obtained by applying to Lemma 13 (18) the automorphism induced by q.

V is also an elementary abelian normal subgroup of G. It is generated by $v_1, v_2, \ldots, v_6$. By definition the elements $v_1, \ldots, v_5$ are contained in G^4.

By Lemma 13 (10),

$$V \leq G^4 .$$

The definition of v_1 shows that V is the normal subgroup in G generated by $(p^2q)^4$.

We shall now describe the normal subgroup generated by $(pq^2)^4$. Let

$$\alpha : G \to G$$

be the involutory automorphism of G which just interchanges p and q. Obviously, (V) is then the normal subgroup in G generated by $(pq^2)^4$. Here are the formulae for the operation of G on V:

$pv_1p^{-1} = v_3$		$qv_1q^{-1} = v_4v_2$
$pv_2p^{-1} = v_1v_2v_3$		$qv_2q^{-1} = v_6v_1$
$pv_3p^{-1} = v_4v_2$		$qv_3q^{-1} = v_4v_5v_6\ v_3$
$pv_4p^{-1} = v_5$		$qv_4q^{-1} = v_4$
$pv_5p^{-1} = v_4$		$qv_5q^{-1} = v_5$
$pv_6p^{-1} = v_4v_5v_6$		$qv_6q^{-1} = v_4v_5v_6$
$av_1a^{-1} = v_2$	$bv_1b^{-1} = v_2v_4v_6$	$cv_1c^{-1} = v_2$
$av_2a^{-1} = v_1$	$bv_2b^{-1} = v_1v_4v_6$	$cv_2c^{-1} = v_1$

$av_3a^{-1} = v_1v_2v_3v_4v_6$ $bv_3b^{-1} = v_1v_2v_3$ $cv_3c^{-1} = v_1v_2v_3v_4v_6$

$av_4a^{-1} = v_4$ $bv_4b^{-1} = v_4$ $cv_4c^{-1} = v_4$

$av_5a^{-1} = v_5$ $bv_5b^{-1} = v_5$ $cv_5c^{-1} = v_5$

$av_6a^{-1} = v_6$ $bv_6b^{-1} = v_6$ $cv_6c^{-1} = v_6$

$dv_1d^{-1} = v_1v_4v_6$ $ev_1e^{-1} = v_1v_5v_6$

$dv_2d^{-1} = v_2v_4v_6$ $ev_2e^{-1} = v_2v_5v_6$

$dv_3d^{-1} = v_3v_5v_6$ $ev_3e^{-1} = v_3v_4v_6$

$dv_4d^{-1} = v_4$ $ev_4e^{-1} = v_4$

$dv_5d^{-1} = v_5$ $ev_5e^{-1} = v_5$

$dv_6d^{-1} = v_6$ $ev_6e^{-1} = v_6$.

To enable a more detailed study of the normal subgroup generated in G by the fourth powers, it will be convenient to somewhat refine some of the formulae, Lemma 2 (36)-(48).

LEMMA 14.

(1) $ece^{-1} = v_3v_4v_6 \cdot bcb^{-1}F$;

(2) $de\ b\ e^{-1}d^{-1} = v_4(cb)^2 \cdot bD^{-1}$;

(3) $ede^{-1}d^{-1} = v_3v_5v_6 \cdot bcb^{-1}c^{-1}D^{-1}$;

(4) $e^2 = v_3v_4v_6bcb^{-1}c^{-1}$.

Proof. (1) By Lemma 2 (43) and the definition of v_3 follows

$$ece^{-1} = bc \cdot v_3 \cdot b^{-1}F .$$

Then Lemma 14 (1) follows by Lemma 13.

(2) By Lemma 2 (41) and Lemma 12,

$$\begin{aligned} de \cdot b \cdot e^{-1}d^{-1} &= c^{-1}b^{-1} \cdot c^{-1}b^{-1}cb^{-1} \cdot ac^{-1}b^{-1}a^{-1}b^{-1} \cdot D^{-1} \\ &= v_6 \cdot c^2bc \cdot abab \cdot cD^{-1} \quad \text{by Lemma 9 (13)} \\ &= v_6 \cdot ccbcb \cdot c^{-1}b \cdot D^{-1} \quad \text{by Lemma 11 (25), (24).} \end{aligned}$$

Then Lemma 14 (2) follows by Lemma 11 (25) and Lemma 13 (19).

(3) By Lemma 2 (45), Lemma 12, Lemma 13,

$$ede^{-1}d^{-1} = v_3v_5v_6babca^{-1}bc^{-1}b^{-1}c\cdot b^2ca\cdot bca^{-1}\cdot bc^{-1}bc\cdot bc\cdot D^{-1}$$

$$= v_3v_6\cdot babca^{-1}b\cdot c^{-1}b^{-1}cb^{-1}\cdot cb^{-1}c^2D^{-1}$$

by Lemma 4 (1), Lemma 11 (24), Lemma 9 (13)

$$= v_3v_5v_6\cdot c^{-1}a^{-1}b^{-1}a^{-1}c^{-1}b^{-1}cb^{-1}cb^{-1}c^2\cdot D^{-1}$$ by Lemma 9 (13), Lemma 11 (25)

$$= v_3v_4v_5\cdot c^{-1}cb\cdot bcb^{-1}c^{-1}\cdot c^2b^{-1}c^2D^{-1}$$ by Lemma 11 (24)

$$= v_3v_4v_6\cdot c^{-1}\cdot bc^{-1}b^{-1}c^{-1}\cdot c^{-1}D^{-1}$$ by Lemma 11 (17), (25).

Then Lemma 14 (3) follows by Lemma 11 (1).

(4) By Lemma 2 (47) and the definition of v_3 follows

$$e^2 = bcv_3b^{-1}c^{-1} .$$

Hence Lemma 14 (4) follows by Lemma 13.

DEFINITION.

$$u_i = \alpha(v_i) , \quad i = 1, 2, \ldots, 6 ,$$

$$U := \langle u_1, \ldots, u_6\rangle .$$

From Lemma 13 it follows that U is an elementary abelian normal subgroup in G which is generated by fourth powers. U is the normal subgroup generated by $(q^2p)^4$, obviously it centralizes V.

LEMMA 15. *The elements* u_i *can be computed in terms of the* v_i *:*

(1) $u_1 = v_6\cdot bFb^{-1}$;

(2) $u_2 = v_6\cdot a^{-1}b^{-1}Fba$;

(3) $u_3 = v_2\cdot cb^2cF$;

(4) $u_4 = v_6\cdot b^{-1}a^{-1}FabF$;

(5) $u_5 = v_4v_5v_6\cdot b^{-1}a^{-1}FabF$;

(6) $u_6 = v_6$;

(7) $u_4u_5 = v_4v_5$.

The following transformation formulae apply:

$$(8)\ \begin{cases} pu_1p^{-1} = u_4u_2 \\ pu_2p^{-1} = u_6u_1 \\ pu_3p^{-1} = u_4u_5u_6u_3 \\ pu_4p^{-1} = u_4 \\ pu_5p^{-1} = u_5 \\ pu_6p^{-1} = u_4u_5u_6 \end{cases} \qquad (9)\ \begin{cases} qu_1q^{-1} = u_3 \\ qu_2q^{-1} = u_1u_2u_3 \\ qu_3q^{-1} = u_4u_2 \\ qu_4q^{-1} = u_5 \\ qu_5q^{-1} = u_4 \\ qu_6q^{-1} = u_4u_5u_6 \ . \end{cases}$$

Proof. The following formulae follow immediately by the definition of $a, b, \ldots, e$ and Lemma 1:

$$\alpha(a) = pqp^{-1}q^{-1} = a^{-1} \ ;$$

$$\alpha(b) = qa^{-1}q^{-1} = ad^{-1} \ ;$$

$$\alpha(c) = q^2a^{-1}q^{-2} = dec \ ;$$

$$\alpha(d) = pq^{-1}p^{-1}\cdot a^{-1} = b^{-1}a^{-1} \ ;$$

$$\alpha(e) = qb^{-1}a^{-1}q^{-1} = abe^{-1}d^{-1} \ ;$$

(1)
$$u_1 = \left(q^2p\right)^4 = p^{-1}Fp = p^{-1}\cdot ebe^{-1}a^{-1}ba^{-1}p$$
$$= dad^{-1}abca^2bc$$

which follows by the definition of F and Lemma 1. Moreover Lemma 2 (36) and Lemma 12 imply

$$u_1 = v_4v_5v_6\cdot D^{-1}babca^2c^2D\cdot D^{-1}cD\cdot Fabca^2bc$$

$\Rightarrow$ $u_1 = v_4v_5v_6\cdot bab\cdot ca^2c^{-1}a^2\cdot a^2b^{-1}a^{-1}Fababca^2bc$ by Lemma 12, Lemma 13 (19)

$\Rightarrow$ $u_1 = v_5\cdot baba^{-1}bca\cdot abcb\cdot Fb^{-1}$ by Lemma 12.

Then Lemma 15 (1) follows by Lemma 11 (24) and Lemma 4 (1).

(2) $\alpha\left(v_2\right) = \alpha\left(av_1a^{-1}\right) = a^{-1}v_6bFb^{-1}a$. Lemma 15 (2) then follows by Lemma 13 and Lemma 12.

(4), (5), (6). Obviously,

$$\alpha\left((ab)^4\right) = D^2 = v_4v_5$$

and

$$\alpha\left((ac)^2\right) = a^{-1}de\cdot ca^{-1}\cdot e^{-1}d^{-1}c \ .$$

By Lemma 2 (44), (38),

$$\alpha\big((ac)^2\big) = a^{-1}Dbc^{-1}bDbabc$$

$$\Rightarrow \qquad \alpha\big((ac)^2\big) = v_4v_5\cdot a^{-1}\cdot bc^{-1}b^{-1}c\cdot c^{-1}b^{-1}abcb^{-1}a^{-1}FabF \quad \text{by Lemma 12}$$

$$\Rightarrow \qquad \alpha\big((ac)^2\big) = v_4v_6\cdot a^{-1}c^2\cdot b^{-1}cab\cdot cb^{-1}a^{-1}FabF \quad \text{by Lemma 11 (17), (25).}$$

Then Lemma 15 (4) follows by Lemma 11 (24), Lemma 4 (1) and Lemma 15 (5) follows by Lemma 11 (26), Lemma 13 (19) and the formula for $\alpha\big((ab)^4\big)$.

$\alpha\big((ac^{-1})^2\big) = a^{-1}c^{-1}da^{-1}c^{-1}d^{-1}$ follows by the definition and $(ed)^2 = 1$ and $ca \rightleftarrows e$. By Lemma 2 (36), (42)

$$\alpha\big((ac^{-1})^2\big) = a^{-1}c^{-1}Dbacb^{-1}D^{-1} .$$

Then Lemma 15 (6) follows by Lemma 11 (24), Lemma 12 and Lemma 13 (19).

(8), (9). These formulae follow by Lemma 13 applying α .

(3) Use Lemma 14 (4) to conclude

$$v_3 = e^2cbc^{-1}b^{-1}v_4v_6 .$$

Apply α , then we have by the statements already proven

$$u_3 = ab\cdot de\cdot abca\cdot e^{-1}d^{-1}\cdot ded^{-1}e^{-1}\cdot ec^{-1}e^{-1}\cdot a^{-1}\cdot u_4\cdot u_6 .$$

By Lemma 14 (1)-(3), Lemma 2 (38), (41), Lemma 13, Lemma 12 follows

$$u_3 = v_1v_2v_6\cdot b^{-1}\cdot D^{-1}cbcb^{-1}c^{-1}a^{-1}\cdot b^{-1}cbc^2b^{-1}a^{-1}Fb^{-1}a^{-1}FabF$$

$$\Rightarrow \qquad u_3 = v_1v_2\cdot b^{-1}D^{-1}cbcb^{-1}a\cdot b^{-1}c^{-1}bc\cdot b^{-1}a^{-1}Fb^{-1}a^{-1}FabFv_5v_6 \quad \text{by Lemma 11 (1), (24)}$$

$$\Rightarrow \qquad u_3 = v_1v_2v_4v_5\cdot b^{-1}cbcb^{-1}a^2b^{-1}c^{-1}D^{-1}c^{-1}D\cdot D^{-1}\cdot Fb^{-1}a^{-1}FabF$$

by Lemma 11 (17), (24), Lemma 12

$$\Rightarrow \qquad u_3 = v_1v_2\cdot b^{-1}cbcb^{-1}a^2b^{-1}c^2ab\cdot v_1\cdot a^{-1}b^{-1}\cdot F$$

by Lemma 12 and the definition of v_1

$$\Rightarrow \qquad u_3 = v_2\cdot b^{-1}cbcb^{-1}ac^{-1}b^{-1}cba^{-1}b^{-1}F \quad \text{by Lemma 11 (24)}$$

$$\Rightarrow \qquad u_3 = v_2v_4v_6b^{-1}cbcb^{-1}cb^{-1}c^{-1}F \quad \text{by Lemma 11 (9)}$$

$$\Rightarrow \qquad u_3 = v_2c^{-1}b^2c^{-1}F \quad \text{by Lemma 11 (1).}$$

Conjugating this relation with c^2 and using Lemma 13, Lemma 15 (7), (8) yields Lemma 15 (3).

By Lemma 15 follows immediately the collection of formulae given hereafter.

The formulae for the operation of G on U

$$\begin{array}{llll}
au_1a^{-1} = u_2 & & bu_1b^{-1} = u_2u_4u_6 \\
au_2a^{-1} = u_1 & & bu_2b^{-1} = u_1u_4u_6 \\
au_3a^{-1} = u_1u_2u_3u_4u_6 & & bu_3b^{-1} = u_1u_2u_3u_4u_5 \\
au_4a^{-1} = u_4 & & bu_4b^{-1} = u_4 \\
au_5a^{-1} = u_5 & & bu_5b^{-1} = u_5 \\
au_6a^{-1} = u_6 & & bu_6b^{-1} = u_6 \\
cu_1c^{-1} = u_2u_4u_5 & & du_1d^{-1} = u_1u_4u_6 \\
cu_2c^{-1} = u_1u_4u_5 & & du_2d^{-1} = u_2u_4u_6 \\
cu_3c^{-1} = u_1u_2u_3u_5u_6 & & du_3d^{-1} = u_3u_4u_6 \\
cu_4c^{-1} = u_4 & & du_4d^{-1} = u_4 \\
cu_5c^{-1} = u_5 & & du_5d^{-1} = u_5 \\
cu_6c^{-1} = u_6 & & du_6d^{-1} = u_6 \\
eu_1e^{-1} = u_1u_5u_6 & & \\
eu_2e^{-1} = u_2u_5u_6 & & \\
eu_3e^{-1} = u_3u_5u_6 & & \\
eu_4e^{-1} = u_4 & & \\
eu_5e^{-1} = u_5 & & \\
eu_6e^{-1} = u_6 \,. & &
\end{array}$$

The join of V and U is again an elementary abelian normal subgroup in G, because U and V centralize each other. Obviously $V \cap U$ is just the group fixed by α. Hence we have

$$\langle v_6,\ v_4 \cdot v_5 \rangle \leq V \cap U \,.$$

It is easy to check that the representation of G on the $\mathbb{F}_2$ vector space $V \cdot U$ is compatible with the relations (i)-(xvi). The factor group $G/V \cdot U$ is of exponent 4 as mentioned in part A. We shall now collect some relations in G/G^4.

LEMMA 16.

(1) $F = u_2u_4$;

(2) $b^2c^2 = u_2u_3v_1v_6$;

(3) $a^2b^2 = u_1u_3u_5v_1v_2v_3v_5$;

(4) $a^2c^2 = u_1u_2u_5u_6v_2v_3v_5$;

(5) $aca^{-1}c^{-1} = u_1u_2u_4v_2v_3v_4$;

(6) $(ab)^2\, d^{-2} = u_1u_3v_1v_3v_4v_5$;

(7) $(ab)^2\, (bc)^{-2} = u_5v_1v_2v_4$;

(8) $d^2\, e^{-2} = u_1u_2u_6$;

(9) $a^2ba^2b^{-1} = u_5u_6v_1v_2$;

(10) $b^2ab^2a^{-1} = u_5v_1v_2v_5$;

(11) $b^2cb^2c^{-1} = u_4u_6v_1v_2$;

(12) $c^2bc^2b^{-1} = u_4v_1v_2v_5$;

(13) $d^{-2}ad^2a^{-1} = u_4u_6$;

(14) $d^{-2}bd^2b^{-1} = u_5u_6$;

(15) $d^{-2}cd^2c^{-1} = u_4u_6$;

(16) $dad^{-1}a^{-1}d^{-2} = u_3v_1v_3$;

(17) $dbd^{-1}b^{-1}d^{-2} = u_2u_4u_6$;

(18) $dcd^{-1}c^{-1}d^{-2} = u_1u_2u_3\, v_1v_3$;

(19) $eae^{-1}a^{-1}d^{-2} = u_1v_4v_5$;

(20) $ebe^{-1}b^{-1}d^{-2} = u_1u_2u_3v_2v_3v_5$;

(21) $ece^{-1}c^{-1}d^{-2} = u_2u_5u_6$;

(22) $da^2d^{-1}a^2 = u_1u_2v_5v_6$;

(23) $db^2d^{-1}b^2 = u_1u_2$;

(24) $dc^2d^{-1}c^2 = u_1u_2u_6v_4$;

(25) $ea^2e^{-1}a^2 = u_1u_2u_5u_6$;

(26) $eb^2e^{-1}b^2 = u_1u_2u_4v_5$;

(27) $ec^2e^{-1}c^2 = u_1u_2u_4u_6$;

(28) $D^2 = v_4v_5$.

Proof. (1) follows by Lemma 15 (1).

(2) follows by Lemma 15 (3), if one replaces F using Lemma 16 (1).

(3) is Lemma 16 (2) transformed by p^{-1} .

(4), (5) follow by Lemma 16 (2), (3).

(6) Use the definition of v_1 to conclude

$$v_1v_4v_6 = aba^{-1}b^2a^{-1}b^{-1}a^{-1}\cdot(ab)^2\cdot D^{-1} .$$

Lemma 16 (6) follows by Lemma 16 (3).

(7) By Lemma 11 (25), (18), (26)

$$(ab)^2\cdot(bc)^{-2} = babacbcb\cdot v_4v_5 .$$

Lemma 16 (7) follows by Lemma 11 (24) and Lemma 16 (2).

(8) Transform Lemma 16 (6) by p and use Lemma 16 (7) to obtain Lemma 16 (8).

(9)-(12) follow by Lemma 16 (2)-(4) under consideration of Lemmas 13 and 15.

(13)-(15) follow by Lemma 16 (6), (7) under consideration of Lemma 11 (25), (26), (22).

(16) By Lemma 2 (36)

$$dad^{-1}a^{-1}D^{-1} = ab\cdot b^{-1}a^{-1}Dba\cdot bca^{-1}\cdot F\cdot bc^{-1}a^{-1}b^{-1}\cdot D^{-1}a^{-1}D\cdot D^{-2} .$$

Lemma 16 (16) follows by Lemma 11 (24), Lemma 16 (1), (13), (3), (4).

(17) By Lemma 2 (39) follows

$$dbd^{-1}b^{-1}D^{-1} = cb \cdot b^2 \cdot F \cdot a^2 \cdot a^{-1}c^{-1}b^{-1} \cdot a \cdot a^2 b^2 \cdot b \cdot D^{-1}b^{-1}D \cdot D^{-2} .$$

Lemma 16 (17) follows by Lemma 11 (24), Lemma 16 (3), (14).

(18) By Lemma 2 (42)

$$dcd^{-1}c^{-1}D^{-1} = ab \cdot b^{-1}a^{-1}Dba \cdot a^{-1}b^{-1} \cdot F \cdot abab \cdot \left(b^{-1}a^{-1}b^{-1}c^{-1}\right)^2 \cdot cD^{-1}c^{-1}D \cdot D^{-2} .$$

Lemma 16 (18) follows by Lemma 11 (25), Lemma 16 (3), (15).

(19)-(21) follow by Lemma 16 (16)-(18) on transformation by p under consideration of Lemma 16 (18).

(22)-(24) follow by raising Lemma 16 (19)-(21) to the second power under consideration of Lemma 12 (10).

REMARK. The formulae in Lemma 16 imply that a^2, b^2 and c^2 commute pairwise.

CHAPTER 2

A FINITENESS PROOF FOR Γ

We shall now study the following group and we show that it is of finite order:

$$
\begin{aligned}
\Gamma = \langle Y, Z \mid \quad & \text{(i)} & & Y^4 = Z^8 = 1 ; \\
& \text{(ii)} & & (YZ^{-1})^2 = 1 ; \\
& \text{(iii)} & & (YZ)^4 = 1 ; \\
& \text{(iv)} & & (Y^2Z^2)^8 = 1 ; \\
& \text{(v)} & & (YZ^2)^8 = 1 ; \\
& \text{(vi)} & & (Z^4Y^2)^8 = 1 ; \\
& \text{(vii)} & & (Z^4YZ^2Y)^8 = 1 ; \\
& \text{(viii)} & & (Y^2ZYZ^{-3})^8 = 1 ; \\
& \text{(ix)} & & (Z^4Y^2Z^2Y^2Z^4Y^2Z^{-2}Y^2)^8 = 1 ; \\
& \text{(x)} & & (Z^4Y^2Z^2Y^2)^8 = 1 ; \\
& \text{(xi)} & & (YZ^2YZ^3Y^2Z^3)^8 = 1 ; \\
& \text{(xii)} & & (Z^3Y)^8 = 1 ; \\
& \text{(xiii)} & & (Z^3Y^2)^8 = 1 ; \\
& \text{(xiv)} & & (Z^4Y^2Z^2Y^2ZY)^8 = 1 ; \\
& \text{(xv)} & & (Z^4Y^2ZY)^8 = 1 ; \\
& \text{(xvi)} & & (Z^4YZY^2)^8 = 1 ; \\
& \text{(xvii)} & & (Z^3Y^2Z^2Y^2)^8 = 1 ; \\
& \text{(xviii)} & & (Z^3Y^2Z^{-2}Y^2)^8 = 1 ; \\
& \text{(xix)} & & (Z^2Y^2ZY)^8 = 1 ; \\
& \text{(xx)} & & (Z^4YZ^2Y^2ZY)^8 = 1 ; \\
& \text{(xxi)} & & (Z^4Y^2Z^2Y^2Z^2Y^2Z^{-2}Y^2Z^4Y^2Z^2Y^2Z^{-2}Y^2Z^{-2}Y^2)^8 = 1 ;
\end{aligned}
$$

$$\text{(xxii)}\quad \left(Z^4Y^2Z^2Y^2Z^2Y^2Z^{-2}Y^2\right)^8 = 1 \ ;$$

$$\text{(xxiii)}\quad \left(Z^4Y^2Z^4Y^2Z^4Y^2Z^{-2}Y^2\right)^8 = 1 \ ;$$

$$\text{(xxiv)}\quad \left(Z^4Y^2Z^4Y^2Z^2Y^2Z^2Y^2\right)^8 = 1 \ ;$$

$$\text{(xxv)}\quad \left(Z^4Y^2Z^{-2}Y^2Z^2Y^2Z^2Y^2\right)^8 = 1\rangle \ .$$

A. The main theorems

We shall carry through our computations in a subgroup of Γ . For this we need the

DEFINITION.

$$p := (ZY)^2Z^{-2} \ ,$$

$$q := (YZ)^2Z^{-2} \ ,$$

$$x := (ZY)^2 \ ,$$

$$\Gamma_1 := \langle p,\ q,\ x\rangle \le \Gamma \ .$$

THEOREM 1. Γ_1 *is a normal subgroup of* Γ *, and we have*

$$|\Gamma : \Gamma_1| = 2^3$$

and

$$\Gamma^4 = \Gamma_1^2 \ .$$

Proof. The formulae in Lemma 1 (Part B) show that Γ_1 is normal in Γ . The quotient Γ/Γ_1 is obviously isomorphic to the dihedral group of order 8 . The presentation of $B(2, 4)$ in Coxeter and Moser (1957, p. 81), proves that Γ^4 is the normal subgroup of Γ , which is generated by Z^4 and $\left(Y^2Z^2\right)^4$. Then Lemmas 1, 3-8 prove $\Gamma^4 = \Gamma_1^2$.

For our further computations, an automorphism of Γ_1 induced by an inner automorphism of Γ will be of importance.

DEFINITION. Let α be the automorphism induced by $ZY^2Z^{-1} = ZYZY^{-1} = xY^2$ on Γ_1 .

THEOREM 2. α *operates on the generators of* Γ_1 *in the following way:*

$$\alpha : p \to q$$
$$q \to p$$
$$x \to x \ .$$

Proof. See the transformation formulae in Lemma 1, Part B.

Next we shall show some relations in Γ_1 between p, q and x. Then we shall find (see Theorem 3 (1)-(15)) that the subgroup of Γ generated by p and q is a factor group of the group studied in Chapter 1. Hence we have in particular the relations shown in Chapter 1, Lemma 16, between p and q. When we obviously use these relations, we shall not indicate it separately.

THEOREM 3. *In* Γ_1 *we have the following relations:*

(1) $p^4 = q^4 = 1$;

(2) $(qp)^4 = 1$;

(3) $(pq^{-1})^4 = 1$;

(4) $(p^2q^2)^4 = 1$;

(5) $(pqp^{-1}q^{-1})^4 = 1$;

(6) $(qp^{-1}qpq^{-1}p)^4 = 1$;

(7) $(p^2qpq^2pq)^2 = 1$;

(8) $(q^2pq^2p^{-1})^8 = 1$, $(p^2qp^2q^{-1})^8 = 1$;

(9) $(p^2q)^8 = (q^2p)^8 = 1$;

(10) $(pqpq^{-1})^8 = (qpqp^{-1})^8 = 1$;

(11) $(p^2qpq^{-1}p^2qp^{-1}q^{-1})^8 = (q^2pqp^{-1}q^2pq^{-1}p^{-1})^8 = 1$;

(12) $(qpq^{-1}p^2)^8 = (pqp^{-1}q^2)^8 = 1$;

(13) $(p^2q^2p^2q^{-1})^8 = (q^2p^2q^2p^{-1})^8 = 1$;

(14) $(pqp^2q^2)^8 = (qpq^2p^2)^8 = 1$;

(15) $(pqp^2q^{-1})^8 = (qpq^2p^{-1})^8 = 1$;

(16) $x^2 = 1$;

(17) $(px)^4 = (qx)^4 = 1$;

(18) $(pq^{-1}x)^2 = 1$;

(19) $(xpxpq^2)^4 = (xqxqp^2)^4 = 1$;

(20) $(pxqx)^4 = 1$;

(21) $(p^{-1}xqx)^4 = 1$;

(22) $(qxqxp)^4 = (pxpxq)^4 = 1$;

(23) $(p^2qx)^4 = (q^2px)^4 = 1$;

(24) $(xpxp^{-1})^4 = (xqxq^{-1})^4 = 1$;

(25) $(q^{-1}p^2qpxpx)^4 = (p^{-1}q^2pqxqx)^4 = 1$;

(26) $(p^2qxqx)^4 = (q^2pxpx)^4 = 1$;

(27) $(qp^2q^{-1}xpxp)^4 = (pq^2p^{-1}xqxq)^4 = 1$;

(28) $(qpq^{-1}xpxp)^4 = (pqp^{-1}xqxq)^4 = 1$;

(29) $(qp^2q^{-1}p^2xp^{-1}x)^4 = (pq^2p^{-1}q^2xq^{-1}x)^4 = 1$;

(30) $(qpq^{-1}pxp^{-1}x)^4 = (pqp^{-1}qxq^{-1}x)^4 = 1$;

(31) $(q^{-1}px)^4 = 1$;

(32) $(xp^2q^2)^4 = 1$, $(q^2p^2x)^4 = 1$;

(33) $(pqpqxqpqpx)^2 = 1$;

(34) $(pqp^{-1}q^{-1}xpxqxp^{-1}x)^4 = (qpq^{-1}p^{-1}xqxpxq^{-1}x)^4 = 1$.

Proof. See Part B.

The list in Theorem 3 is not a complete presentation of Γ_1 , because in our computations we use also the conjugates of relations (1)-(34) under Y or Z .

It is now necessary to pass to a subgroup of Γ_1 .

DEFINITION. Let Γ_2 be the normal subgroup of Γ_1 which is generated by the elements

$$pqp^{-1}q^{-1},\ xpxp,\ xqxq\ .$$

We want to state a system of generators of the group Γ_2 . So assume that

$$
\begin{array}{ll}
a := qpq^{-1}p^{-1} , & s := pxpx , \\
b := pqpq^{-1}p^{-1}p^{-1} , & t := ppxpxp^{-1} , \\
c := p^2qpq^{-1}p^{-1}p^{-2} , & u := p^{-1}qpxpxq^{-1}p , \\
d := q^2pq^{-2}p^{-1} , & v := qpxpxq^{-1} , \\
e := pq^2pq^{-2}p^{-1}p^{-1} , & w := pqpxpxq^{-1}p^{-1} , \\
r := xpxp , & z := p^{-1}q^2pxpxq^{-2}p .
\end{array}
$$

Obviously we have $\langle a, b, c, d, e, r, \ldots, z\rangle \le \Gamma_2$.

THEOREM 4.

$$\Gamma_2 = \langle a, b, c, d, e, r, s, t, u, v, w, z\rangle ;$$

$$|\Gamma_1 : \Gamma_2| \le 2^5 .$$

Proof. Lemma 3 shows

$\langle a, b, c, d, e, r, s, t, u, v, w, z\rangle \trianglelefteq \Gamma_1 \Rightarrow \Gamma_2 \le \langle a, b, c, d, e, r, s, t, u, v, w, z\rangle$.

The other inclusion is obvious. Γ_1/Γ_2 is isomorphic to the semidirect product of an abelian group of order 16 with a group of order 2 .

DEFINITION.

$$
\begin{aligned}
\Gamma_3 &:= \langle a, b, c, d, e, r, s, t, u, v, w\rangle , \\
\Gamma_4 &:= \langle a, b, c, d, e, r, t, u, v, w\rangle , \\
\Gamma_5 &:= \langle a, b, c, d, e, r, t, v, w\rangle , \\
\overline{\Gamma}_4 &:= \langle a, b, c, d, e, s, t, u, v, w\rangle , \\
\overline{\Gamma}_5 &:= \langle a, b, c, d, e, s, t, u, v\rangle .
\end{aligned}
$$

Now we deduce sufficient relations between $a, b, c, d, e, r, s, t, u, v, w, z$ to prove the following theorem.

THEOREM 5. *For the above-defined groups* Γ_i *we have:*

(1) $|\Gamma_2 : \Gamma_3| \le 2$;

(2) $|\Gamma_3 : \Gamma_4| \le 2$;

(3) $|\Gamma_4 : \Gamma_5| \le 2$;

(4) $|\Gamma_3 : \overline{\Gamma}_4| \le 2$;

(5) $|\overline{\Gamma}_4 : \overline{\Gamma}_5| \le 2$.

In particular

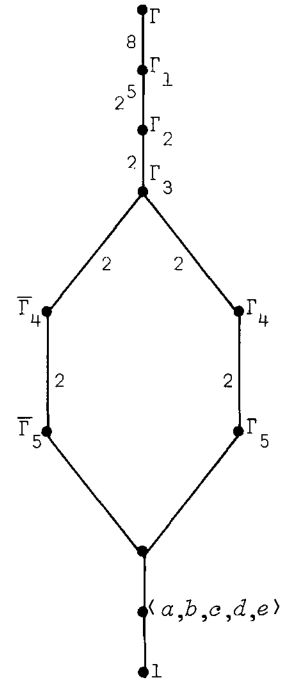
$\Gamma_5 = \langle a, b, c, d, e, r, t, v, w\rangle$

and

$\overline{\Gamma}_5 = \langle a, b, c, d, e, s, t, u, v\rangle$

have finite indices in Γ .

Γ_5 *and* $\overline{\Gamma}_5$ *are subnormal subgroups of* Γ .



Proof. Theorem 5 is proved by Lemmas 8, 9, 10, 9', 10'.

DEFINITION. If G is a group and $S \subset G$ is a subset, then we write $\langle\langle S\rangle\rangle_G$ for the normal subgroup generated by S in G and $C_G(S)$ for the centralizer of S in G .

For our following argumentation the simple theorem stated hereafter is of importance.

THEOREM 6. *Let* G *be a group and* $H_1 \trianglelefteq H \trianglelefteq G$ *where*

(1) H_1 *is finite,*

(2) G/H *is finite;*

then $\langle\langle H_1\rangle\rangle_G$ *is finite.*

Proof. If $g_1, \ldots, g_n$ is a system of coset representatives of G/H , then

$$\langle\langle H_1\rangle\rangle_G = \left\langle g_1H_1g_1^{-1}, \ldots, g_nH_1g_n^{-1}\right\rangle .$$

Since we have

$$g_iH_1g_i^{-1} \trianglelefteq H \quad \text{for all} \quad i = 1, \ldots, n ;$$

hence $\langle\langle H_1\rangle\rangle_G$ is generated by finitely many normal finite subgroups of H .

COROLLARY. *Let* G *be a group with a subnormal subgroup* H *of finite index. Moreover, let* H_1 *be a finite normal subgroup of* H. *Then* $\langle\langle H_1\rangle\rangle_G$ *is also finite.*

Proof. Using Theorem 6 several times yields the corollary.

We repeat again a definition of Chapter 1.

DEFINITION.

$$D := d^2 = \left(q^2pq^{-2}p^{-1}\right)^2 ;$$

$$u_1 := \left(q^2p\right)^4 ;$$

$$u_2 := pqp^{-1}q^{-1}\left(q^2p\right)^4qpq^{-1}p^{-1} ;$$

$$u_3 := q\left(q^2p\right)^4q^{-1} ;$$

$$v_1 := \left(p^2q\right)^4 ;$$

$$v_2 := qpq^{-1}p^{-1}\left(p^2q\right)^4pqp^{-1}q^{-1} ;$$

$$v_3 := p\left(p^2q\right)^4p^{-1} ;$$

$$v_4 := \left(qpq^{-1}p\right)^4 ;$$

$$u_4 := \left(pqp^{-1}q\right)^4 ;$$

$$v_6 := \left(pqp^{-1}q^{-1}p^2qpq^{-1}p\right)^2 ;$$

$$N := \left\langle\left\langle u_1, a^2, de, D\right\rangle\right\rangle_\Gamma .$$

Next we shall show that N is a finite group. In order to prove it, we take elements $\theta \in \langle a, b, c, d, e\rangle$ and show that θ generates a finite normal subgroup either in Γ_5 or in $\overline{\Gamma}_5$. In both cases Theorem 5 and the corollary yield that θ generates also in Γ a finite normal subgroup $\langle\langle\theta\rangle\rangle_\Gamma$. But in the factor group $\Gamma/\langle\langle\theta\rangle\rangle_\Gamma$ the Theorems 1-5 and the Lemmas 1-12 hold. Then, in the factor group, one finds new elements in $\langle a, b, c, d, e\rangle$, whose normal closure in Γ_5 or $\overline{\Gamma}_5$ can be computed. After some steps one exhausts N by this method.

THEOREM 7. N *is a finite group.*

Proof. See Lemmas 13-17.

If we use the same symbols for the images of the elements $a, b, c, d, e, r, s, t, u, v, w$ in Γ/N, then Chapter 1, Lemma 16 yields that $\langle a, b, c, d, e\rangle$ in Γ/N is an elementary abelian group of exponent 2. Moreover, we show in Γ/N that

$$\langle a, b, c, d, e, r, s, t\rangle \trianglelefteq \langle a, b, c, d, e, r, s, t, u, v, w\rangle$$

and that the index is 8 at most.

We use this in Lemma 19. Then Lemma 20 shows that

(1) $$|\langle a, b, c, d, e, r, s, t\rangle : \langle a, b, c, d, e, s, t\rangle| \leq 2 \ ,$$

(2) $$|\langle a, b, c, d, e, s, t\rangle : \langle a, b, c, d, e, s\rangle| \leq 2 \ ,$$

in Γ/N. Hence it suffices to show the finiteness of $\langle a, b, c, d, e, s\rangle$ in Γ/N. We have now deduced so many relations in Γ/N that it is easy to show that

(3) $$\langle a, b, c, d, e, s\rangle \leq \Gamma/N$$

is metabelian at most,

(4) $$\langle a, b, c, d, e, s\rangle \leq \Gamma/N$$

is finite. Thus we have proved

THEOREM 8. *Γ is finite.*

Hence, if we use the shortened form

$$U := YZ^{-1}$$

we see that the group

$$\Gamma^* := \langle U, Y \mid U^2 = Y^4 = (Y^2U)^4 = 1, W^8 = 1\rangle$$

is a factor group of Γ. Hence we get

THEOREM 9. *Γ^* is finite.*

B. The proofs

LEMMA 1. *We have the rules of conjugation*

(1) $YpY^{-1} = qp^{-1}xq^{-1}$,
$YqY^{-1} = xq^{-1}$,
$YxY^{-1} = qp^{-1}x$,

(2) $ZpZ^{-1} = p^{-1}xqx^{-1}p$,
$ZqZ^{-1} = p$,
$ZxZ^{-1} = p^{-1}xq$;

and the formulae

(3) $Z^4 = (xp)^2$;

(4) $(Y^2Z^2)^4 = x(q^{-1}p^{-1})^2x^{-1}$;

(5) $p^2 = ZYZ^4Y^{-1}Z^{-1}$;

(6) $q^2 = YZ^4Y^{-1}$;

(7) $(pq)^2 = ZY(Z^2Y^2)^4Y^{-1}Z^{-1}$;

(8) $(qx)^2 = Y^2Z^4Y^{-2}$.

Proof. All these formulae are obvious consequences of the relations (i)-(iii).

Proof of Theorem 3. By Lemma 1 we have

$$xY^2 : \begin{array}{l} p \to q \\ q \to p \\ x \to x \,. \end{array}$$

Hence, if we have proved one formula of Theorem 3, we have also shown the formula which is implied by the first one when putting q for p and p for q.

In our computations we shall not indicate separately obvious consequences of the relations (i)-(iii).

(1)

$$q^4 = \left(YZYZ^{-1}\right)^4 = YZ^8Y^{-1} = 1$$

by (ii), (i).

(2)

$$\begin{aligned}(pq)^4 &= \left(ZYZYZ^{-2}YZYZ^{-1}\right)^4 \\ &= ZY^{-1}\left(Y^2Z^2\right)^8YZ^{-1} \\ &= 1\end{aligned}$$

by (iv).

(3)

$$\begin{aligned}\left(pq^{-1}\right)^4 &= \left(ZYZYZ^{-1}Y^{-1}Z^{-1}Y^{-1}\right)^4 \\ &= \left(Y^{-1}Z^{-1}Y^{-1}Z^{-1}YZYZ^2YZYZ^{-1}Y^{-1}Z^{-1}Y^{-1}\right)^2 \\ &= \left(Y^2Z\left(YZ^2\right)^4Z^{-1}Y^2\right)^2 \\ &= 1\end{aligned}$$

by (v).

(4)

$$\begin{aligned}\left(p^2q^2\right)^4 &= \left(ZYZYZ^{-1}YZYZ^{-2}YZYZ^{-1}YZYZ^{-1}\right)^4 \\ &= Z\cdot\left(YZ^3YZ^{-2}YZ^3Y\right)^4Z^{-1} \\ &= ZY^{-1}\left(Y^2Z^4\right)^8YZ^{-1} \\ &= 1\end{aligned}$$

by (vi).

(5)

$$\begin{aligned}\left(pqp^{-1}q^{-1}\right)^4 &= \left(ZYZYZ^{-2}YZYZ^2YZYZY^{-1}Z^{-1}Y^{-1}\right)^4 \\ &= \left(ZYZYZ^{-3}Y^{-1}Z^{-1}Y^2Z^{-1}Y^2Z^{-1}Y^{-1}\right)^4 \\ &= \left(ZYZYZ^{-3}Y^2Z^3Y^2\right)^4 = Y^{-1}Z^{-1}Y^{-1}\left(Z^4Y^2\right)^8YZY = 1\end{aligned}$$

by (vi).

(6)

$$
\begin{aligned}
\left(qp^{-1}qpq^{-1}p\right)^4 &= \left(YZYZ^2YZY^2ZY^{-1}Z^{-1}YZYZ^2YZY^2ZY^{-1}Z^{-1}\right)^4 \\
&= \left(YZYZ^2YZY^2ZY^{-1}Z^{-1}\right)^8 \\
&= \left(YZYZ^2YZY^{-1}Z^{-2}\right)^8 \\
&= \left(Z^{-1}Y^{-1}Z^{-2}Y^{-1}Z^{-3}\right)^8 \\
&= Z^{-1}\left(Y^{-1}Z^{-2}Y^{-1}Z^4\right)^8 Z = 1
\end{aligned}
$$

by (vii).

(7) By Lemma 1 we have

$$Y^{-1}pY = qp^2x \ .$$

Hence

$$p\cdot Y^{-1}pY = pqp^2x \ .$$

Moreover, we have

$$Y^2pY^{-1}pYY^{-2} = xqpq^2 \ .$$

We multiply these formulae obtaining

$$
\begin{aligned}
\left(pY^{-1}\right)^8 &= \left(pqp^2qpq^2\right)^2 \ , \\
\left(pY^{-1}\right)^8 &= \left(ZYZYZ^{-2}Y^{-1}\right)^8 \\
&= ZY^{-1}\left(Y^2ZYZ^{-3}\right)^8 YZ^{-1} = 1
\end{aligned}
$$

by (viii).

(8)

$$
\begin{aligned}
q^2pq^2p^{-1} &= YZYZ^{-1}YZY^2ZYZ^{-2}YZYZ^{-1}YZYZ^2YZY \\
&= YZ^3Y^2ZYZ^{-2}YZ^3YZ^2YZY \\
&= YZ^3Y^2Z^2Y^2Z^4YZY^{-1}Z^{-1}Y^{-1}Z^{-1} \\
&= YZ^{-1}Z^4Y^2Z^2Y^2Z^4Y^2Z^{-2}Y^2ZY^{-1} \ .
\end{aligned}
$$

Hence we have $\left(q^2pq^2p^{-1}\right)^8 = 1$ by (ix).

(9)

$$
\begin{aligned}
q^2p &= YZYZ^{-1}YZY^2ZYZ^{-2} \\
&= YZ^3Y^2ZYZ^{-2} \\
&= ZY^{-1}Z^4Y^2Z^2Y^2YZ^{-1} \ .
\end{aligned}
$$

Hence we have $\left(q^2p\right)^8 = 1$ by (x).

(10)

$$\begin{aligned} pqpq^{-1} &= ZYZYZ^{-2}YZY^{2}ZYZ^{-1}Y^{-1}Z^{-1}Y^{-1} \\ &= Y^{-1}Z^{-1}Y^{-1}Z^{-3}Y^{2}Z^{-2}Y^{2}Z^{-1}Y^{2}Z^{2}Y^{2}YZY \\ &= Y^{-1}Z^{-1}Y^{-1}Z^{-3}Y^{2}Z^{-2}Y^{-1}Z^{-2}Y^{-1}YZY \ . \end{aligned}$$

Hence we have $\left(pqpq^{-1}\right)^{8} = 1$ by (xi).

(11)

$$\begin{aligned} \left(q^{2}pqp^{-1}q^{2}pq^{-1}p^{-1}\right)^{8} &= \left(YZ^{3}Y^{2}ZYZ^{-2}YZYZ^{2}YZY^{2}Z^{3}Y^{2}ZYZ^{-1}Y^{-1}Z^{-1}Y^{-1}Z^{3}YZY\right)^{8} \\ &= \left(YZ^{3}Y^{2}Z^{2}Y^{2}Z^{2}Y^{2}Z^{-2}Y^{2}Z^{4}Y^{2}Z^{2}Y^{2}Z^{-1}Y^{-1}Z^{3}YZY\right)^{8} \\ &= YZ^{-1}\left(Z^{4}Y^{2}Z^{2}Y^{2}Z^{2}Y^{2}Z^{-2}Y^{2}Z^{4}Y^{2}Z^{2}Y^{2}Z^{-2}Y^{2}Z^{-2}Y^{2}\right)^{8}ZY^{-1} = \end{aligned}$$

by (xxi).

(12)

$$\begin{aligned} \left(pqp^{-1}q^{2}\right)^{8} &= Z\left(YZYZ^{-2}YZYZ^{2}YZY^{2}Z^{3}Y\right)^{8}Z^{-1} \\ &= ZY\left(Z^{2}Y^{2}Z^{2}Y^{2}Z^{-2}Y^{2}Z^{4}Y^{2}\right)^{8}Y^{-1}Z^{-1} = 1 \end{aligned}$$

by (xxii).

(13)

$$\begin{aligned} \left(p^{2}q^{2}p^{2}q^{-1}\right)^{8} &= \left(ZYZ^{3}YZ^{-2}YZ^{3}Y^{2}Z^{3}YZ^{-1}Y^{-1}Z^{-1}Y^{-1}\right)^{8} \\ &= ZY\left(Z^{4}Y^{2}Z^{4}Y^{2}Z^{4}Y^{2}Z^{-2}Y^{2}\right)^{8}Y^{-1}Z^{-1} = 1 \end{aligned}$$

by (xxiii).

(14)

$$\begin{aligned} \left(pqp^{2}q^{2}\right)^{8} &= \left(ZYZYZ^{-2}YZY^{2}Z^{3}YZ^{-2}YZ^{3}YZ^{-1}\right)^{8} \\ &= ZY\left(Z^{2}Y^{2}Z^{2}Y^{2}Z^{4}Y^{2}Z^{4}Y^{2}\right)^{8}Y^{-1}Z^{-1} = 1 \end{aligned}$$

by (xxiv).

(15)

$$\begin{aligned} \left(qpq^{2}p^{-1}\right)^{8} &= \left(YZY^{2}ZYZ^{-2}YZ^{3}YZ^{2}YZY\right)^{8} \\ &= \left(YZY^{2}Z^{2}Y^{2}Z^{4}YZ^{2}YZY\right)^{8} \\ &= YZ\left(Y^{2}Z^{2}Y^{2}Z^{4}Y^{2}Z^{-2}Y^{2}Z^{2}\right)^{8}Z^{-1}Y^{-1} = 1 \end{aligned}$$

by (xxv).

(16) $x^{2} = (ZYZY)^{2} = (ZY)^{4} = 1$ by (iii).

(17) $Y\ (1)\ Y^{-1} \Rightarrow (17)$.

(18) Y (16) $Y^{-1} \Rightarrow$ (18).

(19)

$$\begin{aligned}(xpxpq^2)^4 &= (ZYZYZYZYZ^{-2}ZYZYZYZYZ^{-2}YZYZ^{-1}YZYZ^{-1})^4 \\ &= (Z^4YZ^3YZ^{-1})^4 \\ &= Z(Z^3Y)^8Z^{-1} = 1\end{aligned}$$

by (xii).

(20) Y (2) $Y^{-1} \Rightarrow$ (20).

(21) Z (3) $Z^{-1} \Rightarrow$ (21).

(22)

$$\begin{aligned}(xpxqp)^4 &= (Z^{-1}YZY^2ZY^2ZYZ^{-2})^4 \\ &= (Z^{-1}Y^2Z^{-3}Y^2Z^{-2})^4 \\ &= Z^{-1}(Y^2Z^{-3})^8Z \\ &= 1\end{aligned}$$

by (xiii).

(23)

$$\begin{aligned}(q^2px)^4 &= (YZ^3Y^2ZYZ^{-1}YZY)^4 \\ &= Y(Z^3Y^2)^8Y^{-1} \\ &= 1\end{aligned}$$

by (xiii).

(24)

$$\begin{aligned}(xpxp^{-1})^4 &= (Z^{-1}YZYZ^3YZY)^4 \\ &= Y^{-1}Z^{-1}(Z^3Y)^8ZY \\ &= 1\end{aligned}$$

by (xii).

(25) We have $ZYp^2qY^{-1}Z^{-1} = xp^{-1}xp^{-1}q^{-1}x$. Hence

$$\begin{aligned}(p^2qZY)^8 &= (p^2qZYp^2qY^{-1}Z^{-1}x)^4 \\ &= (p^2qpxpxq^{-1})^4 \\ &= 1\end{aligned}$$

by (17) and because

$$\begin{aligned}(p^2qZY)^8 &= (ZYZYZ^{-1}YZYZ^{-2}YZY^2)^8\\ &= (ZYZ^4Y^2Z^2Y^2)^8\\ &= 1\end{aligned}$$

by (xiv).

(26) We have $ZY^{-1}p^2YZ^{-1} = xq^{-1}xq^{-1}$. Hence

$$\begin{aligned}(p^2ZY^{-1})^8 &= (p^2ZY^{-1}p^2YZ^{-1})^4\\ &= (p^2qxqx)^4\\ &= 1\end{aligned}$$

because

$$\begin{aligned}(p^2ZY^{-1})^8 &= (ZYZYZ^{-1}YZYZ^{-1}Y^{-1})^8\\ &= (ZYZ^3YZ^{-1}Y^{-1})^8\\ &= (ZYZ^4Y^2)^8\\ &= 1\end{aligned}$$

by (xv).

(27) We have by Lemma 1,

$$\begin{aligned}ZYqp^2Y^{-1}Z^{-1} = q^{-1}p^{-1}xp^{-1} \Rightarrow (qp^2ZY)^8 &= (qp^2ZYqp^2Y^{-1}Z^{-1}x)^4\\ &= (qp^2q^{-1}p^{-1}xp^{-1}x)^4\\ &= 1\end{aligned}$$

because

$$\begin{aligned}(qp^2ZY)^8 &= (YZY^2ZYZ^{-1}YZYZ^{-1}Y)^8\\ &= (YZY^2Z^4)^8\\ &= 1\end{aligned}$$

by (xvi).

(28) Essentially is (22).

(29) We have $ZY^2p^2xY^2Z^{-1} = q^{-1}xp^{-1}$. Hence

$$\begin{aligned}(p^2xZY^2)^8 &= (p^2xZY^2p^2xY^2Z^{-1}ZY^2ZY^2)^4\\ &= (p^2xq^{-1}xp^{-1}ZY^{-1}Z^{-1}Y^{-1})^4\\ &= (p^2xq^{-1}xp^{-1}q^{-1})^4\\ &= 1\end{aligned}$$

because

$$\begin{aligned}(p^2xZY^2)^8 &= (ZYZYZ^{-1}YZYZ^{-1}YZYZY^2)^8\\ &= (ZYZ^5YZY^2)^8\\ &= ZY(Z^{-3}Y^2Z^{-2}Y^2)^8Y^{-1}Z^{-1}\\ &= 1\end{aligned}$$

by (xvii).

(30) We have $ZY^2pxY^2Z^{-1} = p^{-1}xqp$. Hence

$$\begin{aligned}(pxZY^2)^8 &= (pxZY^2pxY^2Z^{-1}ZY^2ZY^2)^4\\ &= (pxp^{-1}xqpZY^{-1}Z^{-1}Y^{-1})^4\\ &= (pxp^{-1}xqpq^{-1})^4\\ &= 1\end{aligned}$$

because

$$\begin{aligned}(pxZY^2)^8 &= (ZYZYZ^{-1}YZYZY^2)^8\\ &= (ZYZ^3YZY^2)^8\\ &= ZY(Z^3Y^2Z^{-2}Y^2)^8Y^{-1}Z^{-1}\\ &= 1\end{aligned}$$

by (xviii).

(31) We have $ZY^{-1}pYZ^{-1} = xq^{-1}$. Hence

$$\begin{aligned}(pZY^{-1})^8 &= (pZY^{-1}pYZ^{-1})^4\\ &= (pxq^{-1})^4\\ &= 1\end{aligned}$$

because

$$\begin{aligned}(pZY^{-1})^8 &= (ZYZYZ^{-1}Y^{-1})^8\\ &= ZY(Z^2Y^2ZY)^8Y^{-1}Z^{-1}\\ &= 1\end{aligned}$$

by (xix).

(32) We have $ZYpq^{-1}Y^{-1}Z^{-1} = xp^{-1}xq$. Hence

$$\begin{aligned}(pq^{-1}ZY)^8 &= (pq^{-1}ZYpq^{-1}Y^{-1}Z^{-1}x)^4\\ &= (pq^{-1}xp^{-1}xqx)^4\\ &= x(qp^2xq)^4x\\ &= 1\end{aligned}$$

because

$$\begin{aligned}(pq^{-1}ZY)^8 &= (ZYZYZ^{-1}Y^{-1}Z^{-1}Y^{-1}ZY)^8\\ &= (ZY^2Z^{-2}Y^2Z^2Y)^8\\ &= ZY^2(Z^{-2}Y^2Z^2YZY^2)^8Y^2Z^{-1}\\ &= ZY^2(Z^{-1}Y^{-1}ZYZ^2YZY^2)^8Y^2Z^{-1}\\ &= ZY^2(Z^{-1}Y^2Z^{-1}Y^{-1}Z^{-1}Y^{-1}ZYZY^2)^8Y^2Z^{-1}\\ &= ZY^2(Z^{-1}Y^2Z^{-1}Y^2YZ^{-1}Z^{-1}Y^2ZY^2)^8Y^2Z^{-1}\\ &= ZY^2(Z^{-1}Y^2Z^{-1}Y^2ZY^{-1}Z^{-1}Y^{-1}Y^{-1}ZY^2)^8Y^2Z^{-1}\\ &= ZY^2(Z^{-1}Y^2Z^{-1}Y^2ZY^{-1}Z^{-1}Y^{-1}Z^{-1}Y^{-1})^8Y^2Z^{-1}\\ &= ZY^2(Z^{-1}Y^2Z^{-1}Y^2Z^2YZ)^8Y^2Z^{-1}\\ &= ZY^2Z^{-1}Y^{-1}Z^{-2}(YZ^{-1}Y^2Z^{-2}Y^2)^8Z^2YZY^2Z^{-1}\\ &= ZY^2Z^{-1}Y^{-1}Z^{-2}(Y^{-1}Z^{-1}Y^2Z^{-2})^8Z^2YZY^2Z^{-1}\\ &= 1\end{aligned}$$

by (xix).

(33) We have $Y^{-1}q^{-1}pY = xpqp^2x$. Hence

$$\begin{aligned}(q^{-1}pY^{-1})^8 &= (q^{-1}pY^{-1}q^{-1}pYY^2q^{-1}pY^{-1}q^{-1}pYY^2)^2\\ &= (q^{-1}pxpqp^2xY^2q^{-1}pxpqp^2xY^2)^2\\ &= (q^{-1}pxpqp^2xxp^{-1}qxqpq^2)^2\\ &= q^{-1}p(xpqpqxqpqp)^2p^{-1}q\\ &= 1\end{aligned}$$

because

$$\begin{aligned}(q^{-1}pY^{-1})^8 &= (ZY^{-1}Z^{-1}Y^{-1}ZYZYZ^{-2}Y^{-1})^8\\ &= YZ^2(Z^4Y^{-1}ZYZY)^8Z^{-2}Y^{-1}\\ &= YZ^2(Z^3Y^2ZY)^8Z^{-2}Y^{-1}\\ &= YZ^2(Z^3Y^{-1}Z^{-1}Y^2)^8Z^{-2}Y^{-1} = 1\end{aligned}$$

by (viii).

(34) We have $ZY^2pqp^{-1}Y^2Z^{-1} = q^{-1}xpxqxp^{-1}xq$. Hence

$$\begin{aligned}(pqp^{-1}ZY^2)^8 &= (pqp^{-1}ZY^2pqp^{-1}Y^2Z^{-1}ZY^2ZY^2)^4\\ &= (pqp^{-1}q^{-1}xpxqxp^{-1}xqq^{-1})^4\\ &= 1\end{aligned}$$

because

$$\begin{aligned}(pqp^{-1}ZY^2)^8 &= (ZYZYZ^{-2}YZYZ^2YZYZY^2)^8 \\ &= Y^{-1}(Z^{-1}Y^{-1}Z^{-3}YZYZY^{-1}Z^{-1})^8Y \\ &= Y^{-1}Z^{-1}(Y^{-1}Z^4Y^{-1}Z^{-1}Y^2Z^{-2})^8ZY \\ &= 1 \text{ by (xx).}\end{aligned}$$

Next we shall express some elements of Γ_2 in terms of $a, b, \ldots, t, z$.

LEMMA 2.

(1) $p^{-1}xpxpp = rst$;

(2) $p^{-1}qxpxq^{-1}p^2 = uc^{-1}b^{-1}a^{-1}vawb$;

(3) $pq^2xpxq^2 = wbtsuzb^{-1}d^{-1}$;

(4) $p^2q^2xpxq^2p^{-1} = zc^{-1}ecvatrva$;

(5) $p^{-1}q^2xpxq^2p^2 = dbruzswb$;

(6) $xqxq = tb^{-1}wa^{-1}vabcuzc^{-1}ec$;

(7) $pxqxqp^{-1} = sabcudbruzswb$;

(8) $p^2xqxqp^2 = zc^{-1}ecvat$;

(9) $p^{-1}xqxqp = dbruz$;

(10) $qxqx = as$;

(11) $qqxqxq^{-1} = da^{-1}v$;

(12) $pqqxqxq^{-1}p^{-1} = eb^{-1}w$.

Proof. (1) follows by using Theorem 3 (17) several times.

(2) follows by using Theorem 3 (17) several times.

(3) By Theorem 3 (3), (17) follows

$$wbtsuzb^{-1}d^{-1} = pq^2(q^{-1}xp^{-1}x)^2(xp^{-1})^2(qpxpx)^2qpq .$$

By Theorem 3 (17), (20), (22) follows (3).

(4) follows obviously by Theorem 3 (20), (21), (17).

(5)

$$\begin{aligned}dbruzswb &= q^{-1}p^{-1}(qpxpx)^3q^2p^2xpxpqpxpxpq^{-1}p^2 \\ &= q^{-1}xpxqp(xp^{-1}xq)^2q^2p^2 \quad \text{by Theorem 3 (17), (22)} \\ &= p^{-1}(xq)^4q^2xpxq^2p^2 \quad \text{by Theorem 3 (21), (18).}\end{aligned}$$

Hence we have shown (5).

(6)

$$
\begin{aligned}
tb^{-1}wa^{-1}vabcuzc^{-1}ec &= p^2xpxpqxp^{-1}xqpxpxpq^2\\
&= p\left(xp^{-1}xq\right)^3q \quad \text{by Theorem 3 (17)}\\
&= pq^{-1}xpxq \quad \text{by Theorem 3 (21)}\\
&= xqxq \quad \text{by Theorem 3 (18).}
\end{aligned}
$$

(7) Use (5) to conclude

$$
\begin{aligned}
sabcudbruzswb &= p(xpxq)^3qp^2\\
&= pq^{-1}xp^{-1}xqp^{-1}p^{-1} \quad \text{by Theorem 3 (20).}
\end{aligned}
$$

Hence we have proved (7).

(8) Use (4) to conclude

$$
\begin{aligned}
zc^{-1}ecvat &= p^2q^{-1}\left(q^{-1}xpx\right)^3p\\
&= p^2q^{-1}xp^{-1}xqp^{-1}p^2 \quad \text{by Theorem 3 (21).}
\end{aligned}
$$

Hence the proof of (8) is complete.

(9) follows obviously by Theorem 3 (22), (18).

(10)

$$
\begin{aligned}
as &= qpq^{-1}xpx\\
&= qxqx \quad \text{by Theorem 3 (18).}
\end{aligned}
$$

(11)

$$
\begin{aligned}
da^{-1}v &= q^2pq^{-1}xpxq^{-1}\\
&= q^2xqxq^{-1} \quad \text{by Theorem 3 (18).}
\end{aligned}
$$

(12)

$$
\begin{aligned}
eb^{-1}w &= pq^2pq^{-1}xpxq^{-1}p^{-1}\\
&= pq^2xqxq^{-1}p^{-1} \quad \text{by Theorem 3 (18).}
\end{aligned}
$$

Next we shall sum up the transformation formulae which are necessary to prove Theorem 4.

LEMMA 3.

(1) $pap^{-1} = b$;

(2) $pbp^{-1} = c$;

(3) $pcp^{-1} = c^{-1}b^{-1}a^{-1}$;

(4) $pdp^{-1} = e$;

(5) $pep^{-1} = b^{-1}db$;

(6) $prp^{-1} = s$;

(7) $psp^{-1} = t$;

(8) $ptp^{-1} = rst$;

(9) $pup^{-1} = v$;

(10) $pvp^{-1} = w$;

(11) $pwp^{-1} = cuc^{-1}b^{-1}a^{-1}vawb$;

(12) $pzp^{-1} = dwbtsuzb^{-1}d^{-1}$;

(13) $qaq^{-1} = da^{-1}$;

(14) $qbq^{-1} = aeb^{-1}a^{-1}$;

(15) $qcq^{-1} = adbc^{-1}b^{-1}a^{-1}$;

(16) $qdq^{-1} = c^{-1}e^{-1}a^{-1}$;

(17) $qeq^{-1} = ad^{-1}abcb^{-1}a^{-1}$;

(18) $qrq^{-1} = abcuc^{-1}b^{-1}a^{-1}$;

(19) $qsq^{-1} = v$;

(20) $qtq^{-1} = awa^{-1}$;

(23) $quq^{-1} = abczc^{-1}b^{-1}a^{-1}$;

(24) $qvq^{-1} = dwbtsuzb^{-1}d^{-1}$;

(25) $qwq^{-1} = aezc^{-1}ecvatrv$;

(26) $qzq^{-1} = abcruzc^{-1}b^{-1}a^{-1}$;

(27) $xax^{-1} = tb^{-1}wa^{-1}vabcuzc^{-1}ecr$;

(28) $xbx^{-1} = rdbruzts$;

(29) $xcx^{-1} = sa^{-1}vc^{-1}e^{-1}czs$;

(30) $xdx^{-1} = ecsvdczuc^{-1}b^{-1}a^{-1}c^{-1}e^{-1}b^{-1}a^{-1}$;

(31) $xex^{-1} = rdbruc^{-1}ecdbdbruzsa^{-1}vabcub^{-1}d^{-1}bcr$;

(32) $xrx^{-1} = s$;

(33) $xsx^{-1} = r$;

(34) $xtx^{-1} = str$;

(35) $xux^{-1} = abcudbruzswbtb^{-1}wa^{-1}vabcuzc^{-1}b^{-1}wszurb^{-1}d^{-1}uc^{-1}b^{-1}a^{-1}$;

(36) $xvx^{-1} = ecsvdwbtsuzrstb^{-1}wd^{-1}vsc^{-1}e^{-1}$;

(37) $xwx^{-1} = rdbruc^{-1}ecvate^{-1}a^{-1}dbruzr$;

(38) $xzx^{-1} = abcudbruzswbd^{-1}abczc^{-1}ecvatrvawbtsuzb^{-1}eb^{-1}wbts$;

(39) $\alpha(a) = a^{-1}$;

(40) $\alpha(b) = ad^{-1}$;

(41) $\alpha(c) = dec$;

(42) $\alpha(d) = b^{-1}a^{-1}$;

(43) $\alpha(e) = abde$;

(44) $\alpha(r) = tb^{-1}wa^{-1}vabcuzc^{-1}ec$;

(45) $\alpha(s) = as$;

(46) $\alpha(t) = da^{-1}v$;

(47) $\alpha(u) = c^{-1}e^{-1}sabcudbruzswbec$;

(48) $\alpha(v) = bt$;

(49) $\alpha(w) = aeb^{-1}wa^{-1}$;

(50) $\alpha(z) = dbaezc^{-1}ecvate^{-1}a^{-1}b^{-1}d^{-1}$.

Proof. (1)-(5) follow by Chapter 1, Lemma 1.

(6)-(12) follow by Lemma 2 (1), (2), (3).

(13)-(17) follow by Chapter 1, Lemma 1.

(18)-(25) follow by Lemma 2 (1), (2), (4).

(26) Because

$$qzq^{-1} = qp^{-1}q^{2}pqp^{-1}(pq^{-1}x)^{2}\ xqxq\ p\ p^{-1}qpq^{-1}$$

this formula is implied by Lemma 2 (9) and Theorem 3.

(27) Theorem 3 (18) yields

$$\begin{aligned} xax &= xqxqp^{-1}xp^{-1}x \\ &= xqxq\ xpxp \quad \text{by Theorem 3 (17).} \end{aligned}$$

Then Lemma 2 (6) implies (27).

(28) By Theorem 3 (18) we have

$$xbx = (xp)^{2}\ p^{-1}(xq)^{2}p\ p^{2}xp^{2}x\ .$$

Then Lemma 2 (2), (9) imply (28).

(29) By Theorem 3 (18) we have

$$xcx = (xp)^{2}\ p^{-1}(xp)^{2}p\ p^{2}(xq)^{2}p^{-2}(px)^{2}\ .$$

Then (29) follows by using Lemma 2 (1), (8).

(30) We write out the right-hand side of the formula obtaining

$$pq^{-1}p^{-1}\ (qpxpx)^{2}\ qpqp^{-1}q^{-1}xpxq^{-1}(px)^{2}\ pqp^{-1}q^{-1}pq^{-1} = xdx$$

using Theorem 3 (17), (18), (22).

(31) By Lemma 2 (5) we have

$$p^{-1}zp = b^{-1}dbdbruzswb \ .$$

Since we have

$$xex = r\, p^{-1}xdxpr \ ,$$

then (31) follows by Lemma 3 (30), (1)-(12) and the formula for $p^{-1}zp$ stated above.

(32)-(34) are obvious.

(35) follows by Lemma 2 (7), (6) and Theorem 3.

(36)

$$ecsvd\ wbtsuzb^{-1}d^{-1}\ db\ rst\ b^{-1}wd^{-1}vsc^{-1}e^{-1}$$
$$= pq^{-1}p^{-1}(qpxpx)^4\ xp^{-1}xp^2\ (qxpx)^2\ qp^{-1}\ \left(q^{-1}pxpx\right)^2\ q^{-1}pqp^{-1}$$

by Lemma 2 (3), (1). Use Theorem 3 (17), (18), (20), (22), to conclude (36).

(37) Use Lemma 2 (9), (8) to obtain (37).

(38) follows immediately by Lemma 2 (3), (4), (7), (12).

(39)-(43) are the formulae contained in Chapter 1, Lemma 15.

(44) follows by Lemma 2 (6).

(45) follows by Lemma 2 (10).

(46) follows by Lemma 2 (11).

(47) follows by Lemma 2 (7).

(48) follows by Theorem 3 (18).

(49) follows by Lemma 2 (12).

(50) follows by Lemma 2 (8).

LEMMA 4. *In addition to the relations proved in Chapter* 1, *we have in the group* $G = \langle p, q\rangle \le \Gamma_1$,

$$(1)\quad u_4 = u_5\ ,\quad (2)\quad v_4 = v_5\ ,\quad (3)\quad D^2 = 1\ .$$

Proof. Theorem 3 (6) yields

$$\left(qp^{-1}qpq^{-1}p\right)^4 = 1\ .$$

Hence

$$(qpc)^4 = 1\ .$$

Now we use the relations of Chapter 1 to obtain (1).

(2) and (3) follow by (1) using Chapter 1, Lemma 16.

Next we collect some relations between the elements a, b, c, d, e, r, s, t, u,

v, w, z which are immediate consequences of Theorem 3.

LEMMA 5.

(1) $u^2 = v^2 = w^2 = r^2 = s^2 = t^2 = z^2 = 1$;

(2) $(uaebc)^2 = 1$;

(3) $(vda^{-1})^2 = 1$;

(4) $(web^{-1})^2 = 1$;

(5) $(rabc)^2 = 1$;

(6) $(sa)^2 = 1$;

(7) $(tb)^2 = 1$;

(8) $(zc^{-1}ecdb)^2 = 1$;

(9) $(rst)^2 = 1$;

(10) $(rt)^2 = 1$;

(11) $(rsts)^2 = 1$;

(12) $(uw)^2 = 1$;

(13) $(cuc^{-1}b^{-1}a^{-1}vawb)^2 = 1$;

(14) $(vcuc^{-1}b^{-1}a^{-1}vawb)^2 = 1$;

(15) $(wbcuc^{-1}b^{-1})^2 = 1$;

(16) $(abcuc^{-1}b^{-1}a^{-1}vawa^{-1}v)^2 = 1$;

(17) $(crstas)^2 = 1$;

(18) $(trabcb)^2 = 1$;

(19) $(ta^{-1}rab^2)^2 = 1$;

(20) $rsa^{-1}bsr = abc^2$;

(21) $stb^{-1}cts = a^{-1}c^{-1}b^{-1}a^{-1}$;

(22) $absbctb^{-1}a^{-1}rstc^{-1}b^{-1}r = 1$;

(23) $arstc^{-1}b^{-1}a^{-1}tcsbr = 1$;

(24) $(vabt)^2 = 1$;

(25) $(bcus)^2 = 1$;

(26) $(ta^{-1}vae)^2 = 1$;

(27) $(sbcuc^{-1}b^{-1}a^{-1}da^{-1})^2 = 1$;

(28) $(tb^{-1}a^{-1}vabcu)^2 = 1$;

(29) $(c^{-1}wbrst)^2 = 1$;

(30) $(crst)^2 = 1$;

(31) $(zr)^2 = 1$;

(32) $(zur)^2 = 1$;

(33) $(dbuc^{-1}b^{-1}a^{-1}vabts)^2 = 1$;

(34) $(ecvawbcrs)^2 = 1$;

(35) $(zc^{-1}ectb^{-1}wa^{-1}vabcu)^2 = 1$;

(36) $(dwbtsuzb^{-1}d^{-1})^2 = 1$;

(37) $(zc^{-1}ecvat)^2 = 1$;

(38) $(zc^{-1}ecvate^{-1}a^{-1})^2 = 1$;

(39) $(dbruz)^2 = 1$;

(40) $(zc^{-1}ecvatrvae)^2 = 1$;

(41) $(zc^{-1}ecvatrvaet)^2 = 1$;

(42) $(zswbe^{-1}a^{-1}dbru)^2 = 1$;

(43) $(zswdbdbru)^2 = 1$;

(44) $(zc^{-1}ecvatrvad^{-1}abc)^2 = 1$;

(45) $(tb^{-1}wa^{-1}vabcuzc^{-1})^2 = 1$;

(46) $b^{-1}d^{-1}seb^{-1}td^{-1}abcrstaebcr = 1$;

(47) $rste^{-1}a^{-1}rda^{-1}sc^{-1}e^{-1}tdec^{-1}e^{-1} = 1$;

(48) $sa^{-1}dstb^{-1}esrc^{-1}b^{-1}dbtsabecr = 1$;

(49) $dbruc^{-1}ecdbc^{-1}ecabcurabc = 1$;

(50) $aeb^{-1}a^{-1}vabcub^{-1}d^{-1}becabcuc^{-1}b^{-1}a^{-1}da^{-1}v = 1$;

(51) $\left(deb^{-1}wbts\right)^2 = 1$.

Proof. (1) Since $u, \ldots, z$ are conjugates of

$$s = (px)^2 ,$$

(1) follows by Theorem 3 (17).

(6)

$$\begin{aligned}(sa)^2 &= \left(pxpxqpq^{-1}p^{-1}\right)^2 \\ &= pqp^{-1}(xq)^4pq^{-1}p^{-1} = 1\end{aligned}$$

by Theorem 3 (17).

(7) p (6) $p^{-1} \Rightarrow$ (7) .

(5) p^2 (7) $p^{-2} \Rightarrow$ (5) .

(3) q (6) $q^{-1} \Rightarrow$ (3) .

(2) p^{-1} (3) $p \Rightarrow$ (2) .

(4) p (3) $p^{-1} \Rightarrow$ (4) .

(8) q (2) $q^{-1} \Rightarrow$ (8) .

(9)

$$\begin{aligned}(rst)^2 &= \left(xpxp\cdot pxpx\cdot p^2xpxp^{-1}\right)^2 \\ &= p^{-1}(xp)^4p = 1\end{aligned}$$

using Theorem 3 (17) several times.

(10) $(rt)^2 = \left(xpxp^{-1}\right)^4 = 1$ by Theorem 3 (24).

(11)

$$\begin{aligned}(rsts)^2 &= \left\{xpxp\cdot pxpx\cdot p^2xp^2x\right\}^2 \\ &= \left(p^{-1}xpx\right)^4 = 1\end{aligned}$$

by Theorem 3 (17), (24).

(12)

$$\begin{aligned}(uw)^2 &= \left(p^{-1}qpxpxq^{-1}p^2qpxpxq^{-1}p^{-1}\right)^2 \\ &= pq\left(q^{-1}p^2qpxpx\right)^4q^{-1}p^{-1} = 1\end{aligned}$$

by Theorem 3 (25).

(14) p (12) $p^{-1} \Rightarrow$ (14).

(15) q (10) $q^{-1} \Rightarrow$ (15).

(13) p (15) $p^{-1} \Rightarrow$ (13).

(16) q (11) $q^{-1} \Rightarrow$ (16).

(17)

$$\begin{aligned}(crstas)^2 &= \left(p^2qxqxp^2qxqx\right)^2 \\ &= \left(p^2qxqx\right)^4 = 1\end{aligned}$$

by Theorem 3 (26).

(18) p (17) $p^{-1} \Rightarrow$ (18).

(19) By Theorem 3 (27) we have

$$\begin{aligned}1 &= \left(qp^2q^{-1}xpxp\right)^4 \\ &= \left(qp^2q^{-1}p^2p^2xpxpp^{-2}p^2qp^2q^{-1}xpxp\right)^2 \\ &= \left(abtb^{-1}a^{-1}r\right)^2 .\end{aligned}$$

Then (19) follows using (7).

(20) By Theorem 3 (28) we have

$$\begin{aligned}1 &= \left(qpq^{-1}p^{-1}pxpxp\right)^4 \\ &= (asp)^4 = aspasp^{-1}\ p^2asp^{-2}\ p^{-1}asp \\ &= asbtcrstc^{-1}b^{-1}a^{-1}r .\end{aligned}$$

Then (20) follows using (5), (6), (7).

(21) p (20) $p^{-1} \Rightarrow$ (21).

(22) By Theorem 3 (29) we have

$$\begin{aligned}1 &= \left(qp^2q^{-1}p^{-2}pxpxp\right)^4 \\ &= (absp)^4 = abs\ pabsp^{-1}\ p^2absp^{-2}p^{-1}absp \\ &= absbctb^{-1}a^{-1}rstc^{-1}b^{-1}r .\end{aligned}$$

Thus we have proved (22).

(23) By Theorem 3 (30) we have

$$\begin{aligned}1 &= (qpq^{-1}pxp^{-1}x)^4 = (pqpq^{-1}p^2xpxpp^{-1})^4 \\ &= (brp^{-1})^4 = br\cdot p^{-1}brp\cdot p^2brp^{-2}pbrp^{-1} \\ &= brarstc^{-1}b^{-1}a^{-1}tcs\ .\end{aligned}$$

Thus we have proved (23).

(24)

$$\begin{aligned}(vabt)^2 &= (q\cdot pxpx\cdot q^{-1}qp^2q^{-1}p^2p^2xpxp^{-1})^2 \\ &= (qxp^{-1}\cdot xpq^{-1}x\cdot pxp^{-1})^2 \\ &= (qxp^{-1})^4 = 1\end{aligned}$$

by Theorem 3 (17), (18), (31).

(25) p^{-1} (24) $p \Rightarrow$ (25).

(26)

$$\begin{aligned}(ta^{-1}vae)^2 &= p^2\cdot(xpxqxpx\cdot q^{-1}p^{-1}q^{-1}p^{-1}\cdot q)^2\cdot p^{-2} \\ &= p^2(xp\cdot xqp^{-1}x\cdot p^{-1}xqpq^2)^2p^{-2} \\ &= p^2(xp^2q^{-1}p^{-1}xqpq^2)^2p^{-2} \\ &= p^2(xp^2q^2)^4p^{-2} = 1\end{aligned}$$

by Theorem 3 (32).

(27) p^{-1} (26) $p \Rightarrow$ (27).

(28)

$$\begin{aligned}(tb^{-1}a^{-1}vabcu)^2 &= (p^2xpxpqp^{-1}xp^2xq^{-1}p)^2 \\ &= p(xq^{-1}p^2)^4p^{-1} = 1\end{aligned}$$

by Theorem 3 (23).

(29) p (24) $p^{-1} \Rightarrow$ (29).

(30) p (7) $p^{-1} \Rightarrow$ (30).

(31) $(zr)^2 = p^{-1}(q^2pxpx)^4p = 1$, by Theorem 3 (26).

(32)

$$\begin{aligned}(zur)^2 &= p^{-1}(q^2pxpx\cdot q^{-1}pxpxq^{-1}pxpx)^2p \\ &= p^{-1}q^{-1}(px)^4qp = 1\end{aligned}$$

by Theorem 3 (22), (17).

(33)

$$\begin{aligned}(dbuc^{-1}b^{-1}a^{-1}vabts)^2 &= (q^{-1}p^{-1}q^2pxpxp^2xpxp^2q^{-1}xp^2x)^2\\ &= (q^{-1}p^{-1}q^2xp^2\cdot xpq^{-1}x\cdot p^2x)^2\\ &= (q^{-1}p^{-1}q^{-1}p^{-1}q^{-1}xpq^{-1}qpqpx)^2\\ &= (pqpqxqpqpx)^2 = 1\ .\end{aligned}$$

By Theorem 3 (17), (18), (33) follows.

(34) p (33) $p^{-1} \Rightarrow$ (34).

(35) Lemma 2 (6) yields $(tb^{-1}wa^{-1}vabcuzc^{-1}ec)^2 = (xq)^4 = 1$, by Theorem 3 (17).

(36) By Lemma 2 (3) we have $(dwbtsuzb^{-1}d^{-1})^2 = q^2(px)^4q^{-2} = 1$ by Theorem 3 (17).

(37) By Lemma 2 (8) we have $(zc^{-1}ecvat)^2 = p^2(xq)^4p^{-2} = 1$ by Theorem 3 (17).

(38) By Lemma 2 (8) we have

$$\begin{aligned}(zc^{-1}ecvate^{-1}a^{-1})^2 &= (p^2xqxpqp)^2\\ &= p^{-1}q^{-1}(xp)^4qp = 1\ .\end{aligned}$$

This follows by Theorem 3 (18), (17).

(39) By Lemma 2 (9) we have $(dbruz)^2 = p^{-1}(xq)^4p = 1$.

(40) By Lemma 2 (4) we have $(zc^{-1}ecvatrvae)^2 = p^2q^2(xp)^4q^2p^2 = 1$.

(41) By Lemma 2 (4) we have

$$(zc^{-1}ecvatrvaet)^2 = p^2(q^2xpxp)^4p^{-2} = 1\ .$$

This last equation follows by Theorem 3 (19).

(42) By Lemma 2 (5) we have

$$\begin{aligned}(dbruzswbe^{-1}a^{-1})^2 &= p^{-1}q^{-1}p^{-1}(pq^{-1}xpxq)^2pqp\\ &= p^{-1}q^{-1}p^{-1}(xq)^4pqp\\ &= 1\ .\end{aligned}$$

We use Theorem 3 (18), (17) for this computation.

(43) By Lemma 2 (5) we have

$$\begin{aligned}(dbruzswbb^{-1}db)^2 &= p^{-1}q^2(xp)^4q^{-2}p\\ &= 1\end{aligned}$$

which is implied using Theorem 3 (17).

(44) By Lemma 2 (4) we have

$$\begin{aligned}(zc^{-1}ecvatrvad^{-1}abc)^2 &= p^2q^{-1}p^{-1}(pq^{-1}xpxq)^2pqp^2 \\ &= p^2q^{-1}p^{-1}(xq)^4pqp^2 \\ &= 1 .\end{aligned}$$

We have used Theorem 3 (17), (18) for this computation.

(45) By Lemma 2 (6) we have

$$\begin{aligned}(tb^{-1}wa^{-1}vabcuzc^{-1}ecc^{-1}e^{-1})^2 &= pq^{-1}(qp^{-1}xqxp)^2qp^{-1} \\ &= pq^{-1}(xp)^4qp^{-1} \\ &= 1\end{aligned}$$

using Theorem 3 (17), (18).

(46) By Lemma 2 (5) we have

$$c^{-1}e^{-1}cz\ dbruzswb = q^2xp^2xq^2p^2 .$$

The 39th formula of this lemma yields

$$c^{-1}e^{-1}curb^{-1}d^{-1}swb = q^2xp^2xq^2p^2 .$$

We apply p^2 to this formula obtaining

$$e^{-1}wtd^{-1}abcrstuc^{-1}b^{-1}a^{-1} = p^2q^2xp^2xq^2 .$$

Multiplying both formulae yields

$$c^{-1}e^{-1}curb^{-1}d^{-1}swbe^{-1}wtd^{-1}abcrstuc^{-1}b^{-1}a^{-1} = 1 .$$

Use Lemma 5 (4), (2) together with the formula $ecae^{-1}a^{-1}c^{-1} = 1$ which is contained in Chapter 1, Lemma 16, to obtain the formula (46).

(47) We apply p^{-1} to (46) obtaining

$$rste^{-1}a^{-1}rda^{-1}sc^{-1}e^{-1}tc^{-1}b^{-1}a^{-1}dab = 1 ,$$

and by Chapter 1, Lemma 16,

$$c^{-1}b^{-1}a^{-1}dabece^{-1}d^{-1} = 1 .$$

(48) By Theorem 3 (34) we have

$$1 = (qpq^{-1}p^{-1}xqxpxq^{-1}x)^4 = q^{-1}(q^2pq^2p^{-1}\cdot pxpx\cdot pqxqx)^4q$$

which yields $(sa^{-1}dsp)^4 = 1$. Hence

$$sa^{-1}dspsa^{-1}dsp^{-1}p^2sa^{-1}dsp^{-2}p^{-1}sa^{-1}dsp = 1 .$$

We use the formulae for p to conclude (48).

(49) By definition we have

$$
\begin{aligned}
dbruc^{-1}ecdbc^{-1}ecabcurabc &= q^{-1}p^{-1}\ qpxpxqpxpx\ qpqp^{-1}q^{-1}\ pq^{-1}x\ pxq^{-1}p\ xpxp\ qp^{-1}q^{-1}p \\
&= q^{-1}\ p^{-1}xp^{-1}x\ p^{-1}q^{-1}xp^{-1}\ xqp^{-1}\ q^{-1}xqxq^{-1}\ xq^{-1}xq^{-1}p \\
&= q^{-1}xpxq^{-1}xq^{-1}xq^{-1}xq^{2}xp \\
&= \left(q^{-1}xp\right)^{2} = 1\ .
\end{aligned}
$$

In this computation Theorem 3 (17), (18), (22) are used.

(50) By definition we have

$$
\begin{aligned}
&aeb^{-1}a^{-1}vabcub^{-1}d^{-1}becabcuc^{-1}b^{-1}a^{-1}da^{-1}v \\
&\qquad = p^{-1}q^{-1}p^{-1}q^{2}p^{-1}xp^{2}xq^{2}pqpqpq^{-1}p^{-1}q^{-1}p^{-1}q^{2}xpxpqxqxq^{-1} \\
&\qquad = qpqpq^{-1}p^{-1}xp^{2}xqp^{-1}q^{2}p^{-1}q^{-1}p^{-1}q^{-1}p^{-1}xqxpqxqxq^{-1} \\
&\qquad = qpqxqp^{-1}xp^{-1}xp^{2}xqp^{-1}q^{-1}pqxqxpqxqxq^{-1} \\
&\qquad = q(pqxqx)^{4}q^{-1} = 1\ .
\end{aligned}
$$

In this computation Theorem 3 (17), (18), (22) are used.

(51)

$$
\begin{aligned}
deb^{-1}wbts &= q^{2}p^{2}q^{-1}xpxpq^{-1}xp^{2}x \\
&= q^{2}p^{2}q^{-1}xpqpx \\
&= \left(q^{2}px\right)^{2}\ .
\end{aligned}
$$

Hence (51) follows by Theorem 3 (23).

Lemma 6 contains some immediate consequences of Lemma 5.

LEMMA 6.

(1) $taet = b^{-1}a^{-1}vaeb^{-1}va$;

(2) $secs = a^{-1}vad^{-1}c^{-1}e^{-1}d^{-1}v$;

(3) $rar = b^{-1}sc^{-1}tabcts$;

(4) $rbr = abcbtc^{-1}b^{-1}a^{-1}t$;

(5) $rcr = stc^{-1}ts$;

(6) $td^{-1}abct = wa^{-1}b^{-1}d^{-1}c^{-1}b^{-1}wb$;

(7) $zaez = c^{-1}ecbe^{-1}a^{-1}vaba^{-1}vc^{-1}e^{-1}c$;

(8) $sDs = vDv$;

(9) $svawbs = a^{-1}b^{-1}wa^{-1}va$;

(10) $ra^{-1}vabcur = abcuc^{-1}b^{-1}a^{-1}vac^{-1}b^{-1}a^{-1}$;

(11) $(wbcuc^{-1}b^{-1})^2 = 1$;

(12) $(vawa^{-1})^4 = 1$;

(13) $(vabcuc^{-1}b^{-1}a^{-1})^4 = 1$;

(14) $(st)^4 = 1$;

(15) $(rs)^4 = 1$.

Proof. (1) By Lemma 5 (7), (26)

$$tb^{-1}a^{-1}vaeta^{-1}vaeb^{-1} = 1 .$$

Lemma 5 (24) yields (1).

(2) We apply α on Lemma 5 (21) obtaining (2).

(3) is Lemma 5 (23).

(4) is Lemma 5 (18) together with Lemma 5 (5).

(5) is Lemma 5 (30).

(6) We apply p to (2) obtaining

$$td^{-1}abct = wec^{-1}b^{-1}a^{-1}deb^{-1}wb ,$$

and by Chapter 1 we have $c^{-1}e^{-1}d^{-1}abce^{-1}a^{-1}b^{-1}d^{-1} = 1$.

(7) By Lemma 5 (37), (38)

$$ze^{-1}a^{-1}z = c^{-1}ecvataeta^{-1}vc^{-1}e^{-1}c .$$

Then (1) implies (7).

(8) By (2) we have

$$sa^{-1}ecs = vad^{-1}c^{-1}e^{-1}d^{-1}v \Rightarrow s(a^{-1}ec)^2s = v(ad^{-1}c^{-1}e^{-1}d^{-1})^2v .$$

By Chapter 1 we then have

$$(a^{-1}ec)^2 = D ,$$

$$(ad^{-1}c^{-1}e^{-1}d^{-1})^2 = D .$$

(9) We apply p^{-1} to Lemma 5 (28) obtaining

$$(asvawb)^2 = 1$$

which proves (9).

(10) We apply p^2 to Lemma 5 (28) obtaining

$$(rabcuc^{-1}b^{-1}a^{-1}va)^2 = 1 \ .$$

(11) is Lemma 5 (15).

(14) By Lemma 5 (9), (10), (11)

$$trst = sr \Rightarrow t(rs)^2t = (sr)^2 \ .$$

On the other hand we have

$$tsrst = srs \Rightarrow t(rs)^2t = (rs)^2 \Rightarrow (rs)^4 = 1 \ .$$

We apply p . Then (14) follows.

(15) is also proved by this procedure.

(12) q (14) $q^{-1} \Rightarrow$ (12).

(13) p (12) $p^{-1} \Rightarrow$ (13).

Next we show the formulae which are necessary to prove Theorems 5 and 6.

LEMMA 7.

(1) $rar = b^{-1}sc^{-1}tabcts$;

(2) $rbr = abcbtc^{-1}b^{-1}a^{-1}t$;

(3) $rcr = abcasbs$;

(4) $$rdr = stc^{-1}b^{-1}a^{-1}tcsbc^{-1}b^{-1}a^{-1}se^{-1}btsd^{-1}ab^{-1} sa^{-1}c^{-1}b^{-1}a^{-1}stece^{-1}d^{-1}tecsad^{-1}b^{-1}sc^{-1} tabctstabctb^{-1}c^{-1}b^{-1}a^{-1}stc^{-1}b^{-1}a^{-1}tcsbtstabctb^{-1}c^{-1}b^{-1}a^{-1} \ ;$$

(5) $rer = stc^{-1}b^{-1}a^{-1}tcsbda^{-1}sc^{-1}e^{-1}tdec^{-1}e^{-1}ts$;

(6) $rtr = t$;

(7) $rsr = tst$;

(8) r *normalizes* $\langle a, b, c, d, e, s, t\rangle$;

(9) u *normalizes* $\langle a, b, c, d, e, v, w\rangle$.

Proof. (1) is Lemma 6 (3).

(2) is Lemma 6 (4).

(3) follows by Lemma 5 (20), (5), (6).

(5) By Lemma 5 (47)

$$ts\ re^{-1}a^{-1}r\ da^{-1}sc^{-1}e^{-1}tdec^{-1}e^{-1} = 1 \ .$$

Using only the first formula of this lemma we obtain (5).

(6) is Lemma 5 (10).

(7) By Lemma 5 (9)

$$1 = rstrst = rsrtst$$

because of Lemma 5 (10).

(4) By Lemma 5 (48)

$$rc^{-1}b^{-1}dbtsabecr = se^{-1}btsd^{-1}as \ .$$

Using (1), (2), (3), (5), (6), (7) we obtain (4).

(9) Obviously the transformation formula in Lemma 3 yields

$$c^{-1}b^{-1}a^{-1}qrq^{-1}abc = u \ ,$$

$$c^{-1}b^{-1}a^{-1}q\langle a, b, c, d, e, s, t\rangle q^{-1}abc = \langle a, b, c, d, e, v, w\rangle$$

such that by (8) we have also proved (9).

Lemma 7' follows in the same way as Lemma 7.

LEMMA 7'.

(1) t *normalizes* $\langle a, b, c, d, e, r, s\rangle$;

(2) w *normalizes* $\langle a, b, c, d, e, u, v\rangle$.

Proof. (1) It is obvious that we must prove

$$tat,\ tbt,\ tct,\ tdt,\ tet,\ trt,\ tst \in \langle a, b, c, d, e, r, s\rangle$$

to show (1):

$tbt \in \langle a, b, c, d, e, r, s\rangle$ is clear by Lemma 5 (7);

$tct \in \langle a, b, c, d, e, r, s\rangle$ follows by Lemma 5 (21);

$tat \in \langle a, b, c, d, e, r, s\rangle$ follows by Lemma 5 (22);

$trt \in \langle a, b, c, d, e, r, s\rangle$ follows by Lemma 5 (10);

$tst \in \langle a, b, c, d, e, r, s\rangle$ is Lemma 7 (7).

By Lemma 5 (46)

$$td^{-1}abct \in \langle a, b, c, d, e, r, s\rangle \Rightarrow tdt \in \langle a, b, c, d, e, r, s\rangle \ .$$

By Lemma 5 (48)

$$tb^{-1}esrc^{-1}b^{-1}dbt \in \langle a, b, c, d, e, r, s\rangle \Rightarrow tet \in \langle a, b, c, d, e, r, s\rangle$$

So we have shown (1).

(2) By Lemma 3 we have

$$a^{-1}qtq^{-1}a = w \ ,$$

$$a^{-1}q\langle a,\ b,\ c,\ d,\ e,\ r,\ s\rangle q^{-1}a = \langle a,\ b,\ c,\ d,\ e,\ u,\ v\rangle \ .$$

We recall the definitions

$$\Gamma_3 = \langle a,\ b,\ c,\ d,\ e,\ r,\ s,\ t,\ u,\ v,\ w\rangle \ ,$$

$$\Gamma_4 = \langle a,\ b,\ c,\ d,\ e,\ r,\ t,\ u,\ v,\ w\rangle \ ,$$

$$\Gamma_5 = \langle a,\ b,\ c,\ d,\ e,\ r,\ t,\ v,\ w\rangle \ .$$

LEMMA 8.

(1) $zaz = rc^{-1}ecvate^{-1}a^{-1}vrtb^{-1}a^{-1}vaeb^{-1}c^{-1}e^{-1}c$
$urb^{-1}d^{-1}ruc^{-1}ecdbrvaete^{-1}a^{-1}vrta^{-1}vc^{-1}e^{-1}c$;

(2) $zbz = urb^{-1}d^{-1}b^{-1}d^{-1}swdbaeb^{-1}wsdbruc^{-1}ecbe^{-1}a^{-1}vaba^{-1}vc^{-1}e^{-1}c$;

(3) $zcz = urb^{-1}d^{-1}ruc^{-1}ecdbuc^{-1}b^{-1}a^{-1}vawbtc^{-1}e^{-1}tb^{-1}wa^{-1}vabcu$;

(4) $zdz = urb^{-1}d^{-1}ruc^{-1}ecvab^{-1}a^{-1}vaeb^{-1}c^{-1}e^{-1}cur$
$b^{-1}d^{-1}swbe^{-1}a^{-1}b^{-1}d^{-1}wsdbdbru$;

(5) $zez = c^{-1}ecvtrvaete^{-1}a^{-1}vrb^{-1}d^{-1}c^{-1}e^{-1}curbd$
$ruc^{-1}ecbe^{-1}a^{-1}vabtrvaeta^{-1}vc^{-1}e^{-1}crc^{-1}ecbe^{-1}a^{-1}vaba^{-1}vc^{-1}e^{-1}c$;

(6) $zrz = r$;

(7) $zsz = urb^{-1}d^{-1}b^{-1}d^{-1}wsdbrstb^{-1}wd^{-1}sdw$
$brstb^{-1}d^{-1}wsrstc^{-1}b^{-1}wdbc^{-1}wbtsrdbru$;

(8) $ztz = c^{-1}ecvatrvaete^{-1}a^{-1}vrta^{-1}vc^{-1}e^{-1}c$;

(9) $zuz = rur$;

(10) $zvz = rc^{-1}ecvate^{-1}a^{-1}vrtb^{-1}a^{-1}vaeb^{-1}c^{-1}e^{-1}c$;

(11) $zwz = urb^{-1}d^{-1}rstb^{-1}wd^{-1}sdwbrstb^{-1}d^{-1}wsrstc^{-1}b^{-1}wdbc^{-1}wbtsrdbru$;

(12) z *normalizes* Γ_3 .

Proof. The formulae (1)-(11) are implied by the formulae Lemma 5 (35)-(45). The formulae (6) and (9) are implied by Lemma 5 (31), (32). Then we have, by Lemma 5 (39),

(**) $$zdbz = urb^{-1}d^{-1}ru$$

and by Lemma 5 (8),

(***) $$zc^{-1}ecz = b^{-1}d^{-1}c^{-1}e^{-1}curdbru \ .$$

By Lemma 5 (35), (45)

$$zc^{-1}c^{-1}e^{-1}cz = uc^{-1}b^{-1}a^{-1}vcawbtectb^{-1}wa^{-1}vabcu \ .$$

The third formula of this lemma is obtained by (***).

By Lemma 5 (43), using (**) and Lemma 5 (39),

(****) $$zswz = urb^{-1}d^{-1}b^{-1}d^{-1}wsdbdbru \ .$$

By Lemma 5 (42)

$$zswzzbzze^{-1}a^{-1}zzdbruz = urb^{-1}d^{-1}aeb^{-1}ws \ .$$

Then (****) and Lemma 5 (39), Lemma 6 (7) imply (2).

We use (2) and (**) obtaining immediately (4).

Use Lemma 5 (40) to obtain

$$zc^{-1}ecvatzzrzzvzzaez = e^{-1}a^{-1}vrta^{-1}vc^{-1}e^{-1}c \ .$$

Then Lemma 5 (31), (37) and Lemma 6 (7) yield (10).

Obviously (8) follows by Lemma 5 (40), (41).

By Lemma 5 (37)

$$zc^{-1}eczzvzzazztz = ta^{-1}vc^{-1}e^{-1}c \ .$$

Use (***), (10), (8) to conclude (1).

We apply p to (10) obtaining (11). Then (****) and (11) yield (7).

Hence we have shown all formulae (1)-(11).

We obtain (12) when we keep in mind that Γ_3 is generated by the elements a, b, c, d, e, r, s, t, u, v, w .

(5) follows by Lemma 6 (7) and Lemma 8 (1).

LEMMA 9.

(1) $sas \in \Gamma_4$;

(2) $sbs \in \Gamma_4$;

(3) $scs \in \Gamma_4$;

(4) $sds \in \Gamma_4$;

(5) $ses \in \Gamma_4$;

(6) $srs \in \Gamma_4$;

(7) $sts \in \Gamma_4$;

(8) $sus \in \Gamma_4$;

(9) $svs \in \Gamma_4$;

(10) $sws \in \Gamma_4$;

(11) s *normalizes* Γ_4 .

Proof. (1) is Lemma 5 (6).

(2) By Lemma 5 (20),

$$sbs = a^{-1}rabc^2r \ .$$

(3) By Lemma 5 (21),

$$scs = rc^2b^{-1}a^{-1}ra^2tc^{-1}tb^{-1}a \ .$$

(8) By Lemma 5 (25),

$$sbcus \in \Gamma_4 \Rightarrow sus \in \Gamma_4$$

using (2), (3).

(4) By Lemma 5 (27),

$$sbcuc^{-1}b^{-1}a^{-1}da^{-1}s \in \Gamma_4 \Rightarrow sds \in \Gamma_4$$

using (1), (2), (3), (8).

(5) follows by Lemma 6 (2) and (3).

(6) By Lemma 6 (9), $svawbs \in \Gamma_4$; by Lemma 5 (34) we have

$$secvawbcrs \in \Gamma_4 \Rightarrow (6)$$

using (3), (5).

(7) By Lemma 5 (9) we have $srts = rt$. Then (7) is implied by (6).

(9) By Lemma 5 (33), $sdbuc^{-1}b^{-1}a^{-1}vabts \in \Gamma_4$. The formulae already proved yield $svs \in \Gamma_4$.

(10) By Lemma 5 (29), $stc^{-1}wbrs \in \Gamma_4$. (1)-(9) yield $sws \in \Gamma_4$.

(11) Γ_4 is generated by $a, b, c, d, e, r, t, u, v, w$.

LEMMA 10.

(1) $uau \in \Gamma_5$;

(2) $ubu \in \Gamma_5$;

(3) $ucu \in \Gamma_5$;

(4) $udu \in \Gamma_5$;

(5) $ueu \in \Gamma_5$;

(6) $uru \in \Gamma_5$;

(7) $utu \in \Gamma_5$;

(8) $uvu \in \Gamma_5$;

(9) $uwu \in \Gamma_5$;

(10) *u normalizes* Γ_5 .

Proof. (1) to (5) follow by Lemma 7 (9). (8) and (9) are also implied by Lemma 7 (9).

(7) By Lemma 5 (28),

$$utb^{-1}a^{-1}vabcu \in \Gamma_5 \Rightarrow utu \in \Gamma_5 .$$

(6) By Lemma 6 (10), $uc^{-1}b^{-1}a^{-1}ra^{-1}vabcu \in \Gamma_5$. The formulae already proved yield $uru \in \Gamma_5$.

(10) By definition Γ_5 is generated by $a, b, c, d, e, r, t, v, w$.

We need another subnormal subgroup of Γ_3 .

DEFINITION.

$$\overline{\Gamma}_4 := \langle a, b, c, d, e, s, t, u, v, w\rangle ,$$

$$\overline{\Gamma}_5 := \langle a, b, c, d, e, s, t, u, v\rangle .$$

Then we obtain

LEMMA 10'.

(1) *r normalizes* $\langle a, b, c, d, e, s, t, u, v, w\rangle = \overline{\Gamma}_4$

(2) *w normalizes* $\langle a, b, c, d, e, s, t, u, v\rangle = \overline{\Gamma}_5$

(3) *$\overline{\Gamma}_5$ is subnormal of finite index in* Γ .

Proof. (1) Lemma 7 shows that

$$rar,\ rbr,\ rcr,\ rdr,\ rer,\ rsr,\ rtr \in \overline{\Gamma}_4 .$$

By Lemma 5 (29)

$$rstc^{-1}wbr \in \overline{\Gamma}_4 \Rightarrow rwr \in \overline{\Gamma}_4 .$$

By Lemma 5 (34)

$$rsecvawbcr \in \overline{\Gamma}_4 \Rightarrow rvr \in \overline{\Gamma}_4 .$$

By Lemma 6 (10)

$$ra^{-1}vabcur \in \overline{\Gamma}_4 \Rightarrow rur \in \overline{\Gamma}_4 .$$

Thus we have shown (1).

(2) By Lemma 7' we have

$$waw,\ wbw,\ wcw,\ wdw,\ wew,\ wuw,\ wvw \in \overline{\Gamma}_5 .$$

By Lemma 6

$$wbsa^{-1}vaw \in \overline{\Gamma}_5 \Rightarrow wsw \in \overline{\Gamma}_5 .$$

By Lemma 5 (51)

$$wbtsdeb^{-1}w \in \overline{\Gamma}_5 \Rightarrow wtw \in \overline{\Gamma}_5 .$$

Thus we have shown (2).

(3) $\overline{\Gamma}_5$ has index 4 in $\overline{\Gamma}_3$. Moreover, $\overline{\Gamma}_5$ is subnormal in $\overline{\Gamma}_3$. Hence follows (3) by the theorems of Part A.

Now we shall construct finite normal subgroups in Γ . To do this we shall make frequent use of the following lemma.

LEMMA 11. *Let* $\theta \in \langle a, b, c, d, e\rangle$ *and* $p^2, q^2, qp, r, \alpha \in C_\Gamma(\theta)$. *Then* $\langle\langle\theta\rangle\rangle_\Gamma$ *is finite. In particular we shall prove step by step:*

(1) $a, b, c, d, e \in C_\Gamma(\theta)$;

(2) $t, v \in C_\Gamma(\theta)$;

(3) $waw, wbw, wcw, wdw, wew \in C_\Gamma(\theta)$;

(4) $uau, ubu, ucu, udu, ueu \in C_\Gamma(\theta)$;

(5) $wvu \in C_\Gamma(\theta)$;

(6) $wvavw$, $wvbvw$, $wvcvw$, $wvdvw$, $wvevw \in C_\Gamma(\theta)$;

(7) $uvavu$, $uvbvu$, $uvcvu$, $uvdvu$, $uvevu \in C_\Gamma(\theta)$;

(8) $urvu$, $utvu$, $urtu \in C_\Gamma(\theta)$;

(9) $wrvw$, $wtvw$, $wrtw \in C_\Gamma(\theta)$;

(10) $wvwawvw$, $wvwbwvw$, $wvwcwvw$, $wvwdwvw$, $wvwewvw \in C_\Gamma(\theta)$;

(11) $(wv)^4 \in C_\Gamma(\theta)$;

(12) $(wr)^4 \in C_\Gamma(\theta)$;

(13) $(wt)^4 \in C_\Gamma(\theta)$.

Proof. (1) By definition we have

$$\langle a, b, c, d, e\rangle \leq \langle p^2, q^2, qp\rangle \leq C_\Gamma(\theta) .$$

(2) Lemma 3 yields

$$p^2rp^{-2} = t ,$$

$$qprp^{-1}q^{-1} = v .$$

(3) By Lemma 5 (6), (20), (21) we have

$$sas = a^{-1} ,$$

$$sbs = a^{-1}rabc^2r ,$$

$$scs = rc^2b^{-1}a^{-1}ra^2tc^{-1}tb^{-1}a .$$

Hence we have obviously

$$sas,\ sbs,\ scs \in C_\Gamma(\theta) .$$

We apply α to these formulae obtaining

$$sds,\ ses \in C_\Gamma(\theta) .$$

But by assumption we have

$$a^{-1}qp \in C_\Gamma(\theta) .$$

On the other hand $a^{-1}qp$ normalizes the group $\langle a, b, c, d, e\rangle$ and we have

$$a^{-1}qpsp^{-1}q^{-1}a = w .$$

Hence we have shown (3).

(4) p^2 normalizes the group $\langle a, b, c, d, e\rangle$ and we have

$$p^2wp^{-2} = u .$$

So we have obviously shown (4).

(5) We have

$$p^2vp^{-2} = cuc^{-1}b^{-1}a^{-1}vawb \in C_\Gamma(\theta) .$$

Then (5) follows by (3), (4).

(6), (7) follow obviously by (3), (4) and (5).

(8) By Lemma 6 (10) we have

$$ra^{-1}vabcurabca^{-1}vabcuc^{-1}b^{-1}a^{-1} = 1 \in C_\Gamma(\theta) .$$

By (1) to (7)

$$urvu \in C_\Gamma(\theta) \Rightarrow urw\cdot wvu \in C_\Gamma(\theta) \Rightarrow urw \in C_\Gamma(\theta) .$$

We apply p^2 obtaining

$$wtu \in C_\Gamma(\theta) \Rightarrow uvtu \in C_\Gamma(\theta) \Rightarrow urtu \in C_\Gamma(\theta) .$$

(9) Applying p^2 to $urtu \in C_\Gamma(\theta)$ yields $wrtw \in C_\Gamma(\theta)$. Moreover we have by (8),

$$urvu = uvwwvrwwvu \in C_\Gamma(\theta) \Rightarrow wvrw \in C_\Gamma(\theta)$$

which suffices to prove (9).

(10) First we show the formulae

$$wuauw,\ wubuw,\ wucuw,\ wuduw,\ wueuw \in C_\Gamma(\theta) .$$

By Lemma 5 (2), (4), (12)

$$wuaebcuw,\ wueb^{-1}uw \in C_\Gamma(\theta) .$$

But by Lemma 5 (15),

$$\left(wbcuc^{-1}b^{-1}\right)^2 = 1 \in C_\Gamma(\theta) \Rightarrow wbcuwuc^{-1}b^{-1}wuubcuc^{-1}b^{-1} \in C_\Gamma(\theta) \Rightarrow wubcuw \in C_\Gamma(\theta) .$$

By Lemma 6 (6)

$$\begin{aligned} wa^{-1}b^{-1}d^{-1}c^{-1}b^{-1}w = td^{-1}abcbt \Rightarrow wua^{-1}b^{-1}d^{-1}c^{-1}b^{-1}uw &= uwa^{-1}b^{-1}d^{-1}c^{-1}b^{-1}wu \\ = utd^{-1}abcbtu = utvuuvd^{-1}abcbvuuvtu \in C_\Gamma(\theta) &\Rightarrow wua^{-1}b^{-1}d^{-1}uw \in C_\Gamma(\theta) . \end{aligned}$$

By Lemma 5 (50) we have

$$ub^{-1}d^{-1}becabcu = c^{-1}b^{-1}a^{-1}vabe^{-1}a^{-1}vad^{-1}abc \ .$$

We use the same method as above obtaining by (1) to (6),

$$wub^{-1}d^{-1}becabcuw \in C_\Gamma(\theta) \Rightarrow wub^{-1}d^{-1}uw \in C_\Gamma(\theta)$$

where we use the formula which is implied by Chapter I, Lemma 16,

$$ecae^{-1}a^{-1}c^{-1} = 1 \ .$$

But the elements db, dba, bc, $aebc$, eb^{-1} generate $\langle a, b, c, d, e\rangle$. Now use

$$wvwawvw = wvuwuawwuvw$$

to conclude from $wuawu \in C_\Gamma(\theta)$ that $wvwawvw \in C_\Gamma(\theta)$. Then the formulae already proved yield that the right hand side is contained in $C_\Gamma(\theta)$.

(11) By Lemma 6 (12) we have

$$1 = \left(vawa^{-1}\right)^4 \in C_\Gamma(\theta) \Rightarrow vawa^{-1}vawa^{-1}vawa^{-1}vawa^{-1} \in C_\Gamma(\theta) \ .$$

Obviously, by using (1) to (10) follows

$$(vw)^4 \in C_\Gamma(\theta) \Rightarrow (wv)^4 \in C_\Gamma(\theta)$$

since $v \in C_\Gamma(\theta)$.

(12) First we shall show the formulae

$$uvuauvu,\ uvubuvu,\ uvucuvu,\ uvuduvu,\ uvueuvu \in C_\Gamma(\theta) \ .$$

For $uvuauvu$ proceed in the following way

$$uvuauvu = uvwwuawwwvu \in C_\Gamma(\theta) \ .$$

Then Lemma 6 (13) yields obviously $(vu)^4 \in C_\Gamma(\theta)$. By Lemma 5 (24), (2) we have

$$tabcut = b^{-1}a^{-1}vu^{-1}uc^{-1}b^{-1}a^{-1}vab \Rightarrow (utabcut)^2 = \left(ub^{-1}a^{-1}vu^{-1}uc^{-1}b^{-1}a^{-1}vab\right)^2 \ .$$

Hence the proofs given hitherto yield that the right hand side of this formula is contained in $C_\Gamma(\theta)$. Then

$$utabcututabcut \in C_\Gamma(\theta) \Rightarrow uvabcututabcu \in C_\Gamma(\theta) \ .$$

Then by (7), $(ut)^4 \in C_\Gamma(\theta)$. We apply p^2 obtaining (12).

(13) If we apply α to (11), then we obtain

$$\left(aeb^{-1}wa^{-1}bt\right)^4 \in C_\Gamma(\theta) \Rightarrow (wt)^4 \in C_\Gamma(\theta) \ .$$

Now we shall prove that $\langle\langle\theta\rangle\rangle_\Gamma$ is finite. To do this we shall show first that $\langle\langle\theta\rangle\rangle_{\Gamma_5}$ is of finite order. We claim that the elements θ, $w\theta w$, $vw\theta wv$, $wvw\theta wvw$ are a conjugacy class in Γ_5. Obviously, all four elements are centralized by a, b, c, d, e. Moreover, we have by (1) to (13),

$$\begin{aligned} r\theta r &= \theta\ , & t\theta t &= \theta\ , \\ rw\theta wr &= vw\theta wv\ , & tw\theta wt &= vw\theta wv\ , \\ rvw\theta wvr &= w\theta w\ , & tvw\theta wvt &= w\theta w\ , \\ rwvw\theta wvwr &= wvw\theta wvw\ , & twvw\theta wvwt &= wvw\theta wvw\ , \end{aligned}$$

$$\begin{aligned} v\theta v &= \theta\ , \\ vw\theta wv &= vw\theta wv\ , \\ vvw\theta wvv &= w\theta w\ , \\ vwvw\theta wvwv &= wvw\theta wvw\ . \end{aligned}$$

Obviously w leaves the above four elements invariant so that $\langle\langle\theta\rangle\rangle_{\Gamma_5}$ is generated by θ, $w\theta w$, $vw\theta wv$, $wvw\theta wvw$.

Because of $\theta \in \langle a, b, c, d, e\rangle$, θ is of finite order and the formulae (3), (6), (10) show that $\langle\langle\theta\rangle\rangle_{\Gamma_5}$ is abelian. Hence $\langle\langle\theta\rangle\rangle_{\Gamma_5}$ is finite.

Γ_5 is a subnormal subgroup of finite index in Γ. Then Theorem 6 of the introduction yields the proof of the lemma.

Next we study the automorphism of Γ_1 which is induced by the element $q^{-1}Y$. By this automorphism we shall obtain the elements θ for which we can use Lemma 11.

LEMMA 12.

(1) $q^{-1}Y(p) = p^{-1}x$;

(2) $q^{-1}Y(q) = q^{-1}x$;

(3) $q^{-1}Y(x) = p^{-1}xq$;

(4) $q^{-1}Y\left(pq^{-1}\right) = p^{-1}q$;

(5) $q^{-1}Y\left(q^{-1}p\right) = pq^{-1}$;

(6) $q^{-1}Y\left(p^2\right) = r$;

(7) $q^{-1}Y\left((abc)^2\right) = abc^{-1}b^{-1}a^{-1}dec$;

(8) $q^{-1}Y\left(u_1de^{-1}\right) = (abc)^2$;

(9) $q^{-1}Y(u_1v_1v_6) = u_1v_1v_6$;

(10) $q^{-1}Y(u_3v_3) = u_1u_2u_3v_1v_2v_3$;

(11) $q^{-1}Y(u_2v_2u_4v_4v_6) = u_2v_2u_4v_4v_6$;

(12) $q^{-1}Y(u_4v_4) = u_4v_4$.

Proof. (1) to (6) follow obviously by the transformation formulae in Lemma 1 and by Theorem 3 (18).

(7) By definition we have $(abc)^2 = (qp^{-1}q^{-1}p)^2$; hence

$$q^{-1}Y((abc)^2) = q^{-1}p^2q^2p^2q^{-1} = abc^{-1}b^{-1}a^{-1}dec$$

using the definitions in Chapter I.

(8) By Chapter 1, Lemma 16, we have

$$u_1de^{-1} = c^{-1}b^{-1}a^{-1}\cdot abecd = p^{-1}qpq^{-1}\cdot pq^{-1}\cdot pq^{-1}p^{-1}q\cdot qp^{-1} .$$

Hence

$$\begin{aligned} q^{-1}Y\left(u_1de^{-1}\right) &= qp^2qp^{-1}qp^{-1}q^2p^{-1}q^{-1}p \\ &= (abc)^2 . \end{aligned}$$

(9) By Chapter I, Lemma 16, we have

$$\begin{aligned} u_1v_1v_6 &= q^{-1}pecp^{-1}q\cdot ed^{-1}u_1\cdot c^{-1}b^{-1}a^{-1} \\ &= q^{-1}ppq^{-1}p^{-1}qp^{-1}qpq^{-1}q^{-1}pqp^{-1}qp^{-1} . \end{aligned}$$

Hence

$$\begin{aligned} q^{-1}Y(u_1v_1v_6) &= pq^{-1}p^{-1}q^2p^{-1}qp^{-1}qp^{-1}q^{-1}pqp^{-1}q^{-1}pqp^2q \\ &= (ec)^2c^2b^{-1}a^{-1}dec^{-1}b^{-1}a^{-1}dec \\ &= u_1v_1v_6 \end{aligned}$$

by Chapter I, Lemma 16.

(10) By Chapter I

$$u_3v_3 = q^{-1}p\cdot u_1v_1v_6p^{-1}q ;$$

hence

$$q^{-1}Y(u_3v_3) = pq^{-1}u_1v_1v_6qp^{-1}$$
$$= u_1u_2u_3v_1v_2v_3 \ .$$

(11) By Chapter I

$$u_2v_2u_4v_4v_6 = pq^{-1}u_3v_3qp^{-1} \ ;$$

hence

$$q^{-1}Y(u_2v_2u_4v_4v_6) = p^{-1}qu_1u_2u_3v_1v_2v_3q^{-1}p$$
$$= u_2v_2v_6u_4v_4 \ .$$

(12) By Chapter I

$$u_4v_4 = u_1v_1v_6 \cdot u_3v_3 \cdot u_2v_2u_4v_4v_6 \cdot pq^{-1} \cdot u_1v_1v_6qp^{-1} \ .$$

Hence we have

$$q^{-1}Y(u_4v_4) = u_1v_1v_6u_1u_2u_3v_1v_2v_3u_2v_2u_4v_4v_6 \cdot p^{-1}qu_1v_1v_6q^{-1}p$$
$$= u_4v_4 \ ,$$

by the transformation formulae for the p, q on the u_i, v_i .

The following lemmas will serve to show that

$$N = \langle\langle u_1, a^2, D, de\rangle\rangle_\Gamma$$

is a finite group. In our computations we shall be a bit sloppy using the same notations for elements contained in Γ and elements contained in factor groups of Γ , since all the Lemmas 1 to 12 apply for any factor group of Γ . In the same way, all formulae stated in Chapter I apply for any factor group of Γ . First we define

DEFINITION.

$$N_1 := \langle\langle u_1u_2v_1v_2, u_4, v_6, u_1u_3v_1v_3\rangle\rangle_\Gamma \ .$$

Now Lemma 12 and Lemma 11 imply

LEMMA 13. *N_1 is a finite group.*

Proof. Let us start with the normal subgroup $M_1 = \langle\langle u_4v_4\rangle\rangle_\Gamma$. By Chapter I

$$qp, \ q^2, \ p^2, \ \alpha \in C_\Gamma(u_4v_4) \ .$$

Then using Lemma 12 (6), (12) we have also

$$r \in C_\Gamma(u_4v_4) \ .$$

Hence Lemma 11 implies that M_1 is finite. For the factor group Γ/M_1 we have by Chapter I,

$$q^2,\ p^2,\ qp,\ \alpha \in C_{\Gamma/M_1}(u_1u_2v_1v_2)\ .$$

Then by Lemma 12 (6), (9), (11),

$$r \in C_{\Gamma/M_1}(u_1u_2v_1v_2)\ .$$

Then Lemma 11 yields that

$$M_2 = \langle\langle u_1u_2v_1v_2,\ u_4v_4\rangle\rangle_\Gamma$$

is finite. By Chapter I we have

$$pu_1u_2v_1v_2p^{-1}u_1u_2v_1v_2 = u_4v_6\ .$$

Hence $u_4v_6 \in M_2$. Then, using the transformation formulae of Chapter I, it is easy to see that

$$qp,\ q^2,\ p^2,\ \alpha \in C_{\Gamma/M_2}(u_1u_3v_1v_3)$$

and

$$qp,\ q^2,\ p^2,\ \alpha \in C_{\Gamma/M_2}(u_1u_3v_1v_3v_6)\ .$$

Using Lemma 12 (6), (9), (10) the first formulae yield that

$$r \in C_{\Gamma/M_2}(u_1u_3v_1v_3v_6)\ .$$

Then Lemma 11 shows that

$$M_3 = \langle\langle u_4v_4,\ u_1u_2v_1v_2,\ u_1u_3v_1v_3v_6\rangle\rangle_\Gamma$$

is finite. Since we have also, by Chapter I,

$$pu_1u_3v_1v_3v_6p^{-1}u_1u_2v_1v_2u_1u_3v_1v_3v_6u_4v_4 = v_6\ ;$$

hence the proof of Lemma 13 is complete.

Now we use another method to compute the normal subgroup generated by D.

DEFINITION.

$$N_2 := \langle\langle D,\ u_1u_2v_1v_2,\ u_4,\ v_6,\ u_1u_3v_1v_3\rangle\rangle_\Gamma\ .$$

Then we obtain by Lemma 13

LEMMA 14. *N_2 is a finite group.*

Proof. Obviously we need only show that $\langle\langle D\rangle\rangle_{\Gamma/N_1} \leq \Gamma/N_1$ is finite. Use Chapter I in Γ/N_1 to obtain $a, b, c, d, e \leftrightarrows D$. By Lemma 6 (8) we have

(*) $$sDs = vDv \ .$$

By Chapter I, Lemma 16 we have $\alpha(D) = D$ in Γ/N_1 . We apply α to the formula (*) obtaining

(**) $$sDs = tDt \ .$$

Applying to this formula $p^{-1}q$, we obtain

(***) $$uDu = c^{-1}b^{-1}a^{-1}vDvabc \ .$$

Now we claim that

(****) $$a, b, c, d, e \rightleftarrows sDs \ .$$

The formula $a \rightleftarrows sDs$ follows by Lemma 5 (6), $b \rightleftarrows sDs$ follows by (**) and Lemma 5 (7).

Apply α to the formula $b \rightleftarrows sDs$ which yields $d \rightleftarrows sDs$. Now (***) becomes

$$cuDuc^{-1} = sDs \ .$$

Then by Lemma 5 (2),

$$caeb \rightleftarrows sDs \Rightarrow cae \rightleftarrows sDs \Rightarrow eca \rightleftarrows sDs$$

by the formula contained in Chapter I, Lemma 16:

$$ecae^{-1}a^{-1}c^{-1} = 1 \Rightarrow ec \rightleftarrows sDs \ .$$

Apply α to this to obtain

$$abc \rightleftarrows sDs \Rightarrow c \rightleftarrows sDs \Rightarrow e \rightleftarrows sDs \ .$$

Hence we have proved (****).

Now (***) reduces to

(***) $$sDs = uDu \ .$$

Of course, these computations apply only in Γ/N_1 !

The formulae (*)-(****) yield clearly that $\{D, sDs\}$ is a conjugacy class in $\overline{\Gamma}_5$. Since $D \in \langle a, b, c, d\rangle$, (****) yields that the group generated by D and sDs is abelian. Hence the normal subgroup generated by D in $\overline{\Gamma}_5$ is finite. Then Lemmas 9' and 10' (3) and the corollary contained in Part A show the lemma.

On our way to the normal subgroup N , next we come to study the normal subgroup

$$N_3 := \langle\langle u_3v_3, D, u_4, v_6, u_1u_2, u_1u_3v_1v_3\rangle\rangle_\Gamma$$

where we can use the same method as in Lemma 13 for N_1 .

LEMMA 15. *N_3 is a finite group.*

Proof. Clearly it suffices to show that $\langle\langle u_3v_3\rangle\rangle_{\Gamma/N_2} \leq \Gamma/N_2$ is finite. In fact, by Chapter I, we have

$$u_1u_2 = DpDp^{-1}v_6 \in N_2 .$$

Now we have in Γ/N_2 ,

$$qp,\ p^2,\ q^2,\ \alpha \in C_{\Gamma/N_2}(u_3v_3) ,$$

such that Lemma 12 (10), (6) force $r \in C_{\Gamma/N_2}(u_3v_3)$.

Using Lemma 11, $\langle\langle u_3v_3\rangle\rangle_{\Gamma/N_2}$ is then a finite group. This completes the proof of Lemma 15.

For our next step the method used in Lemma 14 is again of importance. Let

$$N_4 := \langle\langle u_1,\ N_3\rangle\rangle_\Gamma .$$

LEMMA 16.

(1) $(abcsbs)^2 \in N_3$;

(2) $(secsd^{-1}a)^2 \in N_3$;

(3) $(vavdec)^2 \quad N_3$;

(4) N_4 *is a finite group.*

Proof. (1) We consider the automorphism induced by xY^2pZ on $\langle p,\ q,\ x\rangle$, for which we have, by Lemma 1,

$$\begin{aligned} xY^2pZ \ :\ p &\longmapsto xpx \\ q &\longmapsto q \\ x &\longmapsto xpq^{-1} . \end{aligned}$$

By definition $abcb^{-1} = qp^{-1}q^{-1}p^{-1}qp^{-1}q^{-1}p^{-1}$. We apply again the automorphism using Theorem 3 to compute

$$\begin{aligned} xY^2pZ\left(abcb^{-1}\right) &= qxp^{-1}xq^{-1}xp^{-1}xqxp^{-1}xq^{-1}xp^{-1}x \\ &= qxp^{-1}xp^{-1}xqp^2xqxp^{-1}xq^{-1}xq^{-1}xpq^{-1} \\ &= q\cdot pxpqp^2xqxp^{-1}qxqpq^{-1} \\ &= qp^{-1}q^{-1}x\cdot xqp^2xpqp^2xqxp^{-1}q\cdot xqpq^{-1} \\ &= qp^{-1}q^{-1}xpq^{-1}\cdot xp^{-1}xpqp^{-1}q^{-1}xpxp^{-1}qxqpq^{-1} \\ &= qp^{-1}q^{-1}xpq^{-1}\cdot xp^{-1}xp^{-1}\cdot p^2qp^{-1}q^{-1}p^{-1}pxpxp^{-1}qpq^{-1}\cdot qp^{-1}xqpq^{-1} . \end{aligned}$$

Hence we have

$$qp^{-1}xqpq^{-1}xY^2pZ\left(abcb^{-1}\right) = sb^{-1}sc^{-1}b^{-1}a^{-1} .$$

The proof of (1) is complete.

(2) Apply α to (1) to obtain (2). We must keep in mind that α is induced by an inner automorphism xY^2.

(3) Insert in (2) the formula stated in Lemma 6 (2) to obtain (3).

(4) First take into consideration that

$$pu_3v_3p^{-1} = u_3v_2v_4v_6 \in N_3 \Rightarrow v_2v_3 \in N_3 \ !$$

Then we have in Γ/N_3 by Chapter I, Lemma 16,

$$(*) \qquad aa^{-1}eca^{-1}c^{-1}e^{-1}a = u_1 .$$

By Lemma 6 (2)

$$sa^{-1}ecs = vad^{-1}c^{-1}e^{-1}d^{-1}v .$$

Now apply s to (*), then

$$\begin{aligned} su_1s &= sas\cdot sa^{-1}ecssa^{-1}ssc^{-1}e^{-1}as \\ &= a^{-1}vad^{-1}c^{-1}e^{-1}d^{-1}vavdecda^{-1}v \\ &= a^{-1}vad^{-1}c^{-1}e^{-1}d^{-1}c^{-1}e^{-1}d^{-1}va^{-1}vda^{-1}v . \end{aligned}$$

This holds in Γ/N_3 by (3). Moreover, by Lemma 5 (3),

$$su_1s = a^{-1}vad^{-1}c^{-1}e^{-1}d^{-1}c^{-1}e^{-1}d^{-1}vd^{-1}$$

in Γ/N_3. Then follows in Γ/N_3,

$$\begin{aligned} su_1s &= vad^{-1}c^{-1}e^{-1}d^{-1}c^{-1}e^{-1}d^{-1}vda^{-1} \\ &= vad^{-1}c^{-1}e^{-1}d^{-1}c^{-1}e^{-1}d^{-1}ad^{-1}v . \end{aligned}$$

The formulae contained in Chapter I, Lemma 16, imply

(1) $su_1s = vu_1v$ in Γ/N_3 .

Apply α to this formula obtaining

(2) $su_1s = tu_1t$.

p (2) p^{-1} implies

(3) $su_1s = ru_1r$.

Now we claim

(4) $a, b, c, d, e \rightleftarrows su_1s$.

$a \rightleftarrows su_1s$ follows immediately by Lemma 5 (6); $b \rightleftarrows su_1s$ follows by (2) and Lemma 5 (7); $c \rightleftarrows su_1s$ follows by (3) and Lemma 5 (5).

Apply α to the second and third formula, then $d \rightleftarrows su_1s$ and $e \rightleftarrows su_1s$.

Evidently all these computations apply only in Γ/N_3 !

We apply q to (2) keeping in mind (1) and (4). Then we obtain

(5) $su_1s = wu_1w$.

p^2 (5) p^{-2} implies

(6) $su_1s = uu_1u$.

Obviously the formulae (1) to (6) imply that

$$\{u_1, su_1s\}$$

is a conjugacy class in Γ_3 . Since $u_1 \in \langle a, b, c, d, e\rangle$, $\langle\langle u_1\rangle\rangle_{\Gamma_3}.N_3/N_3$ is an abelian group of order 4 at most.

Since Γ_3 is subnormal of finite index in Γ , our Lemma 16 follows.

Now we have gathered enough formulae together to take one of the last steps showing that

$$N := \left\langle\left\langle a^2, de, D, u_1\right\rangle\right\rangle_\Gamma = \left\langle\left\langle a^2, de, N_4\right\rangle\right\rangle_\Gamma$$

is finite.

LEMMA 17. *N is a finite group.*

Proof. Again it will be sufficient to show that $\langle\langle a^2, de\rangle\rangle_{\Gamma/N_4}$ is finite. Bu now it suffices to show that $\langle\langle a^2\rangle\rangle_{\Gamma/N_4}$ is finite, since, by Lemma 12 (8),

$$u_1de^{-1} = Y^{-1}q(abc)^2q^{-1}Y \ ;$$

that is $de \in \left\langle\left\langle a^2,\ N_4\right\rangle\right\rangle_{\Gamma}$.

By Lemma 5 (6) we have, obviously, $sa^2s = a^2$. But in Γ/N_4 , a^2 is centralized by p and q (by Chapter I, Lemma 16!). Since r, t, u, v, w are all conjugates of s under p, q , we have

$$\begin{aligned} a^2 &\in \operatorname{center}\left(\langle a, b, c, d, e, r, s, t, u, v, w\rangle \cdot N_4/N_4\right) \\ &= \operatorname{center}\left(\Gamma_3 \cdot N_4/N_4\right) . \end{aligned}$$

Hence $\langle\langle a^2\rangle\rangle_{\Gamma_3N_4/N_4}$ is finite. Since Γ_3 is subnormal of finite index in Γ , our lemma follows.

In order to show the finiteness of Γ , we must collect some relations in Γ/N , where we shall also use the same notations for the elements in Γ and their images in Γ/N . For these computations remember that $\langle a, b, c, d, e\rangle$ in Γ/N is elementary abelian of exponent 2 and that it is generated by a, b, c, d .

LEMMA 18. *In* Γ/N *we have the relations*

(1) $(vb)^2 = 1$;

(2) $(tad)^2 = 1$;

(3) $(asr)^2 = 1$;

(4) $(vr)^2 = 1$;

(5) $(ws)^2 = 1$;

(6) $\left(aeb^{-1}ws\right)^2 = 1$;

(7) $vav = atata$;

(8) $\left(c^{-1}b^{-1}a^{-1}da^{-1}vr\right)^2 = 1$;

(9) $(bts)^2 = 1$;

(10) $(vas)^2 = 1$;

(11) $\left(c^{-1}b^{-1}a^{-1}st\right)^2 = 1$;

(12) $(crs)^2 = 1$;

(13) $tact = csbsacsbcs$.

Proof. (1) In Γ/N we have

$$1 = da^{-1}b^{-1}d^{-1}c^{-1}b^{-1}a^{-1}dec = q^2p^2qp^2qp^2q^{-1}p^2q \ .$$

Moreover note that

$$\begin{aligned} q^{-1}Y : p &\mapsto p^{-1}x \\ q &\mapsto q^{-1}x \\ x &\mapsto p^{-1}xq \ . \end{aligned}$$

Hence

$$\begin{aligned} q^{-1}Y(q^2p^2qp^2qp^2q^{-1}p^2q) &= xqx\cdot qxpxpq^{-1}pxpq^{-1}pxpxqxpxpx\cdot xq^{-1}x \\ &= xqx\cdot q(xp)^2q^{-1}pqp^{-1}q^{-1}q(xp)^2q^{-1}qxqx(px)^2\cdot xq^{-1}x \\ &= xqx\cdot abcuc^{-1}b^{-1}a^{-1}a^{-1}\cdot abcuc^{-1}b^{-1}a^{-1}\cdot as\cdot s\cdot xq^{-1}x = 1 \end{aligned}$$

which then implies $uaua = 1$ in Γ/N .

We apply p obtaining (1).

(2) α (1) $\Rightarrow$ (2).

(3)
$$\begin{aligned} ZaZ^{-1} &= Zqpq^{-1}p^{-1}Z^{-1} \\ &= xqxp^{-1}xq^{-1}xp \\ &= q^{-1}\cdot qxqxp^{-1}xp^{-1}xq \\ &= q^{-1}asrq \ . \end{aligned}$$

Then $a^2 = 1$ yields (3) in Γ/N .

(4) We have

$$\begin{aligned} b^{-1}a^{-1} = p^2qp^2q^{-1} \Rightarrow ZYb^{-1}a^{-1}Y^{-1}Z^{-1} &= pxpxq^{-1}xpxpq \\ &= q^{-1}qpxpxq^{-1}xpxpq \\ &= q^{-1}vrq \ . \end{aligned}$$

In Γ/N we have $(ab)^2 = 1$; hence (4).

(5) p (4) $p^{-1} \Rightarrow$ (5) .

(6) By definition

$$q^2p^{-1}q^2p = c^{-1}e^{-1}c \Rightarrow ZYq^2p^{-1}q^2pY^{-1}Z^{-1} = xqxqpqxqxp^{-1}$$
$$= q^{-1}qxqxqpqxqxp^{-1}q^{-1}q$$
$$= q^{-1}asaeb^{-1}a^{-1}awa^{-1}q$$
$$= q^{-1}seb^{-1}wa^{-1}q \ .$$

Since in Γ/N , $(c^{-1}e^{-1}c)^2 = 1$; we get (6).

(7) In Γ/N we have

$$q^2p^2q^{-1}p^2q^2p^2qp^2 = e^{-1}d^{-1}b^{-1}a^{-1}bdea^{-1} = 1 \ .$$

Hence

$$q^{-1}Y(q^2p^2q^{-1}p^2q^2p^2qp^2) = q^{-1}xq^{-1}xp^{-1}xp^{-1}qp^{-1}xp^{-1}xq^{-1}xq^{-1}xp^{-1}xp^{-1}xq^{-1}xp^{-1}xp^{-1}x$$
$$= x(qpq^{-1}p^{-1}q(xp)^2q^{-1}(qx)^2q(xp)^2q^{-1}(px)^2)x$$
$$= xa^2bcuc^{-1}b^{-1}sabcuc^{-1}b^{-1}a^{-1}sx$$
$$= 1$$

which implies

$$a^2bcuc^{-1}b^{-1}sabcuc^{-1}b^{-1}a^{-1}s = 1 \ .$$

We apply p obtaining

$$b^{-1}a^{-1}vab^{-1}ta^{-1}vat = 1 \ .$$

By Lemma 5 (24)

$$ta^{-1}t = ba^{-1}vavab \ .$$

Then (1) and $a^2 = 1$ yield (7).

(8) p^{-1} (6) $p \Rightarrow$ (8).

(9) p (3) $p^{-1} \Rightarrow$ (9).

(10) α (9) $\Rightarrow$ (10).

(11) p^2 (9) $p^{-2} \Rightarrow$ (11).

(12) p (9) $p^{-1} \Rightarrow$ (12).

(13) By (12) follows

$$rsrrcr = c^{-1}s \ .$$

We insert rcr using Lemma 7 (3) obtaining

$$rsr = c^{-1}sbcsbs \ . \tag{*}$$

Lemma 7 (1) yields

$$rasrrsr = b^{-1}sc^{-1}tabcts \ .$$

Use (3) and (*) to conclude (13).

Now we show that it suffices to show the finiteness of the group $\langle a, b, c, d, e, r, s, t\rangle$ in Γ/N .

LEMMA 19. *In* Γ/N *we have*

(1) v *normalizes* $\langle a, b, c, d, e, r, s, t\rangle$ *;*

(2) u, v, w *normalize* $\langle a, b, c, d, e, r, s, t\rangle$ *;*

(3) *in* $\langle a, b, c, d, e, r, s, t, u, v, w\rangle$ *,* $\langle a, b, c, d, e, r, s, t\rangle$ *is of index* 8 *at most.*

Proof. (1) By Lemma 18 (7)

$$vav \in \langle a, b, c, d, e, r, s, t\rangle \ ;$$

$$vbv \in \langle a, b, c, d, e, r, s, t\rangle$$

follows by Lemma 18 (1).

$$vdv \in \langle a, b, c, d, e, r, s, t\rangle$$

follows by Lemma 5 (3),

$$vrv \in \langle a, b, c, d, e, r, s, t\rangle$$

follows by Lemma 18 (4).

Then by Lemma 18 (8),

$$vcv \in \langle a, b, c, d, e, r, s, t\rangle \ .$$

By Lemma 5 (24) we have

$$vabtv = tb^{-1}a^{-1} \ .$$

Hence

$$vtv \in \langle a, b, c, d, e, r, s, t\rangle \ .$$

Moreover, by Lemma 18 (10),

$$vsv \in \langle a, b, c, d, e, r, s, t\rangle \ .$$

Hence we have proved (1) in Γ/N (!) .

(2) We have:

$$u = p^{-1}vp \ ,$$

$$w = pvp^{-1} \ .$$

By Lemma 3, $\langle a, b, c, d, e, r, s, t\rangle$ is normalized by p .

(3) $\langle a, b, c, d, e, r, s, t\rangle$ is a normal subgroup in $\langle a, b, c, d, e, r, s, t, u, v, w\rangle$. But by Lemma 5 (28), (34), (12) we have

$$\left.\begin{matrix}(uv)^2\\ (vw)^2\\ (uw)^2\end{matrix}\right\} \in \langle a, b, c, d, e, r, s, t\rangle .$$

Hence the quotient is elementary abelian. Since u, v, w are involutions, it is of order 8 at most.

Next we show that it is sufficient to show the finiteness of the group $\langle a, b, c, d, e, s\rangle$ in Γ/N in order to prove that Γ is finite.

LEMMA 20. *In* Γ/N *we have*

(1) r *normalizes* $\langle a, b, c, d, e, s, t\rangle$,

(2) t *normalizes* $\langle a, b, c, d, e, s\rangle$,

(3) $\langle a, b, c, d, e, s\rangle$ *is in* $\langle a, b, c, d, e, r, s, t\rangle$ *of index* 4 *at most.*

Proof. (1) is our Lemma 7.

(2) By Lemma 5 (7) we have $tbt \in \langle a, b, c, d, e, s\rangle$. Use Lemma 18 (9) to obtain $tst \in \langle a, b, c, d, e, s\rangle$. By Lemma 5 (21) we have $tct \in \langle a, b, c, d, e, s\rangle$. Use Lemma 18 (13) to conclude $tat \in \langle a, b, c, d, e, s\rangle$. Then Lemma 18 (2) implies $tdt \in \langle a, b, c, d, e, s\rangle$.

Obviously these computations suffice to prove (2).

(3) reformulates (1) and (2).

By the next lemma we shall now prove the finiteness of $\langle a, b, c, d, e, s\rangle$ in Γ/N . Then the preceding lemmas show that the finiteness of $\langle a, b, c, d, e, s\rangle$ implies the finiteness of Γ .

LEMMA 21. *The subgroup* $\langle a, b, c, d, e, s\rangle$ *of* Γ/N *is finite.*

Proof. First we claim that in Γ/N , $a, b, c, d, e \rightleftarrows sbs$, $a, b, c, d, e \rightleftarrows scs$. To show these formulae, it suffices to see that

$$(*) \qquad b, c \rightleftarrows sbs$$

since

$$(**) \qquad a, bcd \in \operatorname{center}\langle a, b, c, d, e, s\rangle)$$

and since a, b, c, d , are involutions! $bcd \in \operatorname{center}(\langle a, b, c, d, e, s\rangle)$ follows by the formula $sbcds = bcd$, which is obtained by applying p^{-1} to Lemma 18 (2). To prove (*) remember that Lemma 5 (21) yields $tct = bsbcs$ in Γ/N . Obviously this formula implies

(***) $$b \rightleftarrows sbcs \; .$$

By Lemma 18 (11) follows $tacttbst = sabc$ which implies $csbsacsbcs.sb = sabc$ by Lemma 18 (9) and (13) which then implies $c \rightleftarrows sbs$ and $b \rightleftarrows scs$. Then, using (***) we have proved (*). Hence we know that $\Delta := \langle b, sbs, c, scs \rangle$ is an abelian subgroup of $\langle a, b, c, d, s \rangle$, which is centralized by a, b, c, d, e . Obviously Δ is also normalized by s . Hence Δ is normal in $\langle a, b, c, d, e, s \rangle$. Δ is of order 16 at most. By (**) the quotient $\langle a, b, c, d, e, s \rangle / \Delta$ is of order 8 at most.

CHAPTER 3

A DESCRIPTION OF G

In this chapter we describe the application of computer implemented algorithms to investigation of the group G, examined in Chapter 1, and some related groups.

All computer calculations were done on a Univac 1100/42 at the Australian National University in Canberra. The principal programs used were an implementation of the nilpotent quotient algorithm (NQA, see Newman (1976)) and implementations of two different versions of coset enumeration, namely the Haselgrove-Leech-Trotter method (HLT) and the Felsch method (see Cannon, Dimino, Havas and Watson (1973)), with certain auxiliary programs. Henceforth reference to the NQA or coset enumeration will include reference to a computer implementation.

First consider the group G (Chapter 1, p. 56). The NQA applied to the presentation for G yields that G has a maximal nilpotent 2-quotient, G_f, of order 2^{21} and class 9. (Of course the proof in Chapter 1 shows $G = G_f$, and in this chapter we describe an independent proof.)

The consistent power commutator presentation for G_f given by that program follows. Detailed information about G_f (hence G) can be read from this presentation.

We denote the initial generators p and q of G by 1 and 2, and the additional generators introduced by 3 to 21. For brevity we omit those power relations which specify that squares of generators are trivial and those commutator relations which specify that a commutator is trivial. For convenient reference we also list the relations defining the additional generators separately.

The relations which define generators 3 to 21 are:

$3 = [2, 1]$, $4 = 1^2$, $5 = 2^2$, $6 = [3, 1]$, $7 = [3, 2]$, $8 = [6, 1]$,
$9 = [6, 2]$, $10 = [7, 2]$, $11 = [8, 1]$, $12 = [8, 2]$, $13 = [9, 1]$,
$14 = [12, 1]$, $15 = [12, 2]$, $16 = [14, 1]$, $17 = [14, 2]$, $18 = [15, 1]$,
$19 = [16, 2]$, $20 = [18, 1]$, $21 = [19, 2]$.

The nontrivial power relations are:

$1^2 = 4$, $2^2 = 5$, $3^2 = 8.9.10.13.15.18.19.21$, $6^2 = 11.14.16.17.18.19$,
$7^2 = 11.15.18.19.20$, $8^2 = 16.17$, $10^2 = 16.17$, $11^2 = 21$, $13^2 = 21$.

The nontrivial commutator relations are:

[2, 1] = 3 , [3, 1] = 6 , [3, 2] = 7 , [4, 2] = 6.8.9.10.11.13.14.15.16.17 ,
[4, 3] = 8.11.14.18.19.21 , [5, 1] = 7.8.9.10.14.17.21 ,
[5, 3] = 10.11.15.16.17.18.19.20.21 , [5, 4] = 9.13.17.21 , [6, 1] = 8 ,
[6, 2] = 9 , [6, 3] = 12.13.15.17.19.20 , [6, 4] = 11.16.17 ,
[6, 5] = 12.13.14.15.18.21 , [7, 1] = 9.12.14.17.20 , [7, 2] = 10 ,
[7, 3] = 13.14.15.17 , [7, 4] = 13.14.16.21 , [7, 5] = 11.18.19.20.21 ,
[7, 6] = 14.15.16.17.20 , [8, 1] = 11 , [8, 2] = 12 , [8, 3] = 14.16.18.19.20.21 ,
[8, 4] = 16.17 , [8, 5] = 15 , [8, 6] = 16.20 , [8, 7] = 18.19 , [9, 1] = 13 ,
[9, 2] = 12.13.14.15.18.21 , [9, 3] = 14.15.16.19.21 , [9, 5] = 16.17.18.20.21 ,
[9, 6] = 16.21 , [9, 7] = 16.17.18.19.21 , [9, 8] = 20.21 ,
[10, 1] = 12.14.15.16.19 , [10, 2] = 11.16.17.18.19.20.21 , [10, 3] = 15.20 ,
[10, 4] = 14.16.18.21 , [10, 5] = 16.17 , [10, 6] = 17.19.20 ,
[10, 7] = 16.17.18.19.20 , [10, 8] = 19.20 , [10, 9] = 19 , [11, 1] = 16.17.21 ,
[11, 2] = 16.17.21 , [11, 4] = 21 , [11, 5] = 21 , [12, 1] = 14 , [12, 2] = 15 ,
[12, 3] = 17.18.19.20 , [12, 4] = 16 , [12, 5] = 16.17.18.19.20 ,
[12, 6] = 19.20.21 , [12, 7] = 19.20 , [12, 8] = 21 , [12, 10] = 21 ,
[13, 1] = 21 , [13, 2] = 15.17.19 , [13, 3] = 18.20 , [13, 5] = 16.17.18.20 ,
[13, 6] = 20.21 , [13, 7] = 20 , [13, 8] = 21 , [13, 9] = 21 , [14, 1] = 16 ,
[14, 2] = 17 , [14, 3] = 19 , [14, 4] = 21 , [14, 5] = 19.21 , [14, 7] = 21 ,
[15, 1] = 18 , [15, 2] = 16.17.18.19.20 , [15, 3] = 20 , [15, 4] = 20 ,
[15, 6] = 21 , [16, 1] = 21 , [16, 2] = 19 , [16, 5] = 21 , [17, 2] = 19.21 ,
[17, 5] = 21 , [18, 1] = 20 , [18, 4] = 21 , [19, 2] = 21 , [20, 1] = 21 .

Further investigation of G_f , using the consistent power commutator presentation and computer collection of eighth powers of an appropriate set of words (see Havas and Newman (these proceedings)), reveals that G_f does indeed have exponent 8 .

The NQA does not cast any light on the question whether G is finite or not. For groups given by presentations a major approach to finiteness questions is coset enumeration. However, the order of the maximal nilpotent quotient of G implies that direct coset enumeration is likely to be very difficult, even if the group is finite. The largest subgroups of G which are easily seen (from the presentation for G on p. 56) to be finite have orders at most 8 . This means enumerating at least 2^{18} cosets over these subgroups to prove finiteness directly. Fortunately an indirect approach is available.

Given a finitely presented group K and a subgroup H generated by a set $\{h_j\}$

the following coset enumeration based technique exists for finding a subgroup H_1, which is a cyclic extension of H (see Havas (1974)). If i is a coset of H such that $i.h = i$ for each element h of the generating set of H then i is called a normalizing (or stabilizing) coset of H. If c_i is a coset representative of the normalizing coset i then $c_i h c_i^{-1} \in H$; so $H_1 = \langle H, c_i \rangle$ is a cyclic extension of H. For our purposes we wish to start with a finite subgroup H of a group K and construct $H_1 = \langle H, c_i \rangle$ with finite order. If c_i is a coset representative for a normalizing coset and m is the least positive integer such that $c_i^m \in H$ (m is the order of c_i modulo H) then $|\langle H, c_i \rangle| = m|H|$.

The implementations of coset enumeration available in Canberra include facilities for finding normalizing cosets, for finding coset representatives and for determining the orders of group elements modulo the subgroup whose cosets are being enumerated. There is no difficulty in completely determining all normalizing cosets, coset representatives and the orders modulo the subgroup of the coset representatives from a complete table of cosets for a subgroup of a group.

What is vital here is that from an incomplete coset table, under favourable circumstances, normalizing cosets (and coset representatives) can be found. Observe that if we find from an incomplete coset table for a subgroup H of a group K that coset i normalizes H, and that $c_i^m \in H$ then, due to possibly undiscovered coincidences between cosets, the actual order modulo H of c_i is a divisor of m. This is enough to ensure that the cyclic extension $H_1 = \langle H, c_i \rangle$ is finite provided H is finite. However, from an incomplete table it is possible that no power of c_i can be found to be in H. Fortunately, when a normalizing coset (and coset representative) can be found from an incomplete table a multiple of the order of the coset representative can frequently also be found. This suffices for our purposes.

This technique can be applied repeatedly provided a suitable (possibly incomplete) coset table can be found, which yields normalizing cosets with representatives of finite order. Under these circumstances we can construct a subnormal chain of subgroups

$$H = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \cdots .$$

Moreover, when H is finite and multiples of the orders (modulo the subgroup being extended) of the extending elements are calculable, then upper bounds for the orders of the subgroups in the chain are calculable.

Let us look at the application of the normalizing coset technique to G. For brevity we include details of only one chain of subgroups constructed by the method. Consider the subgroup H of G generated by $pqpq^{-1}$. From the initial presentation for G, $|H|$ divides 8. An HLT coset enumeration for $G|H$ with a limit of 8000 cosets does not of course complete. However four normalizing cosets are found, cosets with representatives q^2, pq^2p^{-1}, $pq^{-1}pq$ and $q^{-1}p^{-1}qp$ with orders modulo H dividing $2, 2, 8$ and 8, respectively. Let $H_1 = \langle H, pq^{-1}pq \rangle$. Then $|H_1|$ divides 2^6. With a limit of 8000 cosets the HLT enumeration $G|H_1$ does not complete, but three normalizing cosets are found, with representatives pq^{-1}, $q^{-1}p^{-1}$ and q^{-2} which have orders modulo H_1 dividing $4, 4$ and 2 respectively. Let $H_2 = \left\langle H_1, q^{-1}p^{-1} \right\rangle = \langle pq^{-1}, pq \rangle$. Then $|H_2|$ divides 2^8, so $[G : H_2]$ is a multiple of $2^{13} = 8192$. Using the Felsch method of coset enumeration the index $[G : H_2]$ was determined to be 8192 (with a maximum of 8192 cosets). It follows that $|G|$ divides 2^{21}. This information combined with that given by the NQA proves that $|G| = 2^{21}$.

A modification to the NQA yields that the multiplicator of G has rank 7. This implies that 9 is a lower bound for the number of relations required for a presentation of G (see Huppert (1967), p. 631). With this motivation consider the presentation

$$\langle p, q \mid p^4 = q^4 = (pq)^4 = (pq^{-1})^4 = (p^2q^2)^4 = (pqp^{-1}q^{-1})^4 = (p^2qpq^2pq)^2 = (p^2q)^8 = (pq^2)^8 = 1 \rangle$$

(that is, a presentation with the 9 "simplest" relations of G). Clearly the group with this presentation, say G_m, has G as a homomorphic image. The NQA reveals G is the maximal nilpotent quotient of G_m.

If G_m and G are isomorphic then we have found a minimal presentation for G. Since their nilpotent quotients are identical, to prove that G_m and G are isomorphic it suffices to show that G_m is a finite 2-group.

To investigate the finiteness of G_m we cannot follow the same subnormal chain we used in G because we cannot read from the presentation for G_m that the starting point, $pqpq^{-1}$, has finite order. Consider the subgroup H of G_m generated by p^2q. Clearly $|H|$ divides 8. With a limit of $20\,000$ cosets the HLT enumeration $G_m|H$ does not complete, but yields one normalizing coset with representative $pqp^{-1}q^{-1}p^{-2}q^{-1}p^{-2}q^{-1}p^{-1}qp^{-1}$ with order modulo H dividing 2. (An HLT enumeration with an 8000 coset limit does not yield any normalizing cosets.) Let $H_1 = \langle H, pqp^{-1}q^{-1}p^{-2}q^{-1}p^{-2}q^{-1}p^{-1}qp^{-1}\rangle$. Then $|H_1|$ divides 2^4. With a limit of 8000 cosets, the HLT enumeration $G_m|H$ does not complete, but yields one normalizing coset with representative $pqp^{-1}q^{-1}p^{-1}qp^{-1}$ with order modulo H_1 dividing 2. Let $H_2 = \left\langle H_1, pqp^{-1}q^{-1}p^{-1}qp^{-1}\right\rangle$. Then $|H_2|$ divides 2^5. With a limit of 8000 cosets the HLT enumeration $G_m|H_2$ does not complete, but yields one normalizing coset with representative $pq^{-2}p^{-1}$ with order modulo H_2 dividing 2. Let $H_3 = \left\langle H_2, pq^{-2}p^{-1}\right\rangle$. Then $|H_3|$ divides 2^6. Using a limit of 8000 cosets, the HLT enumeration $G_m|H_3$ does not complete, but yields one normalizing coset with representative $pq^{-1}p^{-1}$ with order modulo H_3 dividing 2. Let $H_4 = \left\langle H_3, pq^{-1}p^{-1}\right\rangle = \langle p^2q, pq^{-1}p^{-1}\rangle$. Then $|H_4|$ divides 2^7, so $[G_m : H_4]$ is a multiple of $2^{14} = 16\ 384$ if $G_m = G$. Using the Felsch method of coset enumeration the index $[G_m : H_4]$ was determined to be 16 384 (with a maximum of 16 384 cosets!).

It follows that the nine relator presentation for G_m is a minimal presentation for G.

Observe the effectiveness of the normalizing coset extension method in this case. Each coset representative used actually had as order modulo subgroup the number which was given as a multiple of its order from the incomplete coset table.

CHAPTER 4

A DESCRIPTION OF Γ

In this chapter we describe the application of computer implemented algorithms to the investigation of the group Γ examined in Chapter 2 and some related groups.

The NQA applied to the presentation for Γ (Chapter 2, p. 124) yields that Γ has a maximal nilpotent 2-quotient, Γ_f, of order 2^{33} and class 21 . The consistent power commutator presentation for Γ_f, given by the NQA follows. (The proof in Chapter 2 implies that $\Gamma = \Gamma_f$, so the presentation below gives detailed information about Γ .)

We denote the initial generators Y and Z of Γ by 1 and 2 , and the additional generators introduced by 3 to 33 . For brevity we omit those power relations which specify that squares of generators are trivial and those commutator relations which specify that a commutator is trivial. For convenient reference we also list the relations defining the additional generators separately.

The relations which define generators 3 to 33 are:

$3 = [2. 1]$, $4 = 1^2$, $5 = [3, 1]$, $6 = [3, 2]$, $7 = [5, 1]$, $8 = [7, 2]$,
$9 = [8, 1]$, $10 = [9, 1]$, $11 = [9, 2]$, $12 = [11, 1]$, $13 = [12, 1]$,
$14 = [12, 2]$, $15 = [13, 1]$, $16 = [14, 1]$, $17 = [15, 2]$, $18 = [16, 2]$,
$19 = [17, 1]$, $20 = [19, 1]$, $21 = [19, 2]$, $22 = [20, 1]$, $23 = [21, 1]$,
$24 = [22, 2]$, $25 = [23, 1]$, $26 = [24, 1]$, $27 = [25, 1]$, $28 = [26, 2]$,
$29 = [28, 1]$, $30 = [29, 1]$, $31 = [29, 2]$, $32 = [30, 1]$, $33 = [32, 2]$.

The nontrivial power relations are:

$1^2 = 4$, $2^2 = 3.4.5.6.7.10.11.13.15.17.19.21.23.26.27.29.30.31.32$,

$3^2 = 5.6.10.12.15.17.18.21.22.23.24.25.26.33$,

$5^2 = 11.12.13.17.18.25.26.28.31.32.33$,

$7^2 = 10.15.16.17.18.19.21.22.26.27$,

$8^2 = 14.18.20.23.25.28.30.31.33$,

$9^2 = 16.19.20.23.24.25.27.28.30.33$,

$11^2 = 18.26.27.28.29.30$, $12^2 = 26.27.30.31$, $17^2 = 33$.

The nontrivial commutator relations are:

[2, 1] = 3 , [3, 1] = 5 , [3, 2] = 6 , [4, 2] = 6 ,
[4, 3] = 7.10.11.12.13.15.16.18.19.21.22.25.27.30.32.33 , [5, 1] = 7 ,
[5, 2] = 7.8.9.10.11.16.21.24.25.27.28.29.32.33 ,
[5, 3] = 8.12.13.14.16.17.19.24.25.26.27.28.31.32.33 ,
[5. 4] = 9.12.13.14.16.19.20.24.25.27.29.33 ,
[6, 1] = 7.8.11.13.14.15.18.21.24.25.26.28.30.33 ,
[6, 2] = 7.9.10.11.12.13.14.17.24.27.30.32.33 ,
[6, 3] = 8.12.13.14.16.17.19.24.25.26.27.28.31.32.33 ,
[6, 5] = 10.11.13.15.21.22.23.24.26.27.28.29.30.31.32 ,
[7, 1] = 9.10.12.13.14.15.17.18.20.21.24.28.33 , [7, 2] = 8 ,
[7, 3] = 9.10.12.13.16.17.18.19.20.22.31.33 ,
[7, 4] = 10.13.16.19.20.23.25.26.27.28.32.33 ,
[7, 5] = 10.12.13.16.17.18.19.22.23.29.33 , [7, 6] = 10.13.18.21.23.25.28.30.31.33 ,
[8, 1] = 9 , [8, 2] = 9.14.16.20.21.22.23.25.27.28.29.31.32.33 ,
[8, 3] = 10.11.13.14.16.18.19.21.22.23.24.25.27.29.30.32 ,
[8, 4] = 10.16.19.20.23.24.25.27.28.30.32.33 ,
[8, 5] = 13.15.16.19.21.22.23.24.27.28.29.30.32.33 ,
[8, 6] = 14.17.18.20.21.22.24.26.27.30.31.33 ,
[8, 7] = 13.15.16.17.20.21.25.27.28.33 , [9, 1] = 10 , [9, 2] = 11 ,
[9, 3] = 12.17.19.20.21.23.27.30.31 , [9, 4] = 15.16.19.25.26.27.32 ,
[9, 5] = 13.15.16.17.23.25.28.30.32.33 , [9, 6] = 13.17.21.24.27.29.30.32.33 ,
[9, 7] = 15.20.21.24.26.27.29.31.33 , [9, 8] = 19.20.22.23.24.26.27.28.32 ,
[10, 1] = 15.16.19.25.26.27.32 , [10, 2] = 13.14.15.16.19.20.21.25.26.27.32.33 ,
[10, 3] = 15.16.19.22.24.27.28.29.31.33 , [10, 4] = 27.33 ,
[10, 5] = 19.20.21.22.26.29.31.32 , [10, 6] = 28.30.31.33 , [10, 7] = 22.24.30.32 ,
[10, 8] = 20.27.29.33 , [10, 9] = 22.25.26.27.29.30 , [11, 1] = 12 ,
[11, 2] = 12.17.18.31.32 , [11, 3] = 13.14.18.19.21.22.23.24.26.29.30.32.33 ,
[11, 4] = 13.26.27.30.31 , [11, 5] = 17.19.23.25.27.29.30.32 ,
[11, 6] = 18.20.24.25.26.27.28.29.30 , [11, 7] = 17.19.20.24.26.30 ,
[11, 8] = 24.26.29.31.32.33 , [11, 9] = 21.22.24.26.27.29.30 ,
[11, 10] = 24.25.32.33 , [12, 1] = 13 , [12, 2] = 14 ,
[12, 3] = 15.16.22.24.25.28.29.32.33 , [12, 4] = 15 ,
[12, 5] = 17.19.20.21.23.25.29.30.31.33 , [12, 6] = 17.32.33 ,
[12, 7] = 19.20.21.23.24.26.27.28.32.33 , [12, 8] = 21.22.23.25.29.32 ,
[12, 9] = 23.24.25.27.30.31.33 , [12, 10] = 26.27.30.31 ,
[12, 11] = 29.30.31.32.33 , [13, 1] = 15 ,
[13, 2] = 15.17.18.20.24.25.26.27.28.29.31.32.33 ,
[13, 3] = 17.19.22.24.26.29.30.31 , [13, 4] = 32 ,

[13, 5] = 20.21.24.25.26.28.29.30.31.32 , [13, 6] = 30.33 ,
[13, 7] = 20.22.24.25.31.32.33 , [13, 8] = 24.25.31 , [13, 9] = 25.31.32 ,
[13, 10] = 30.33 , [13, 11] = 33 , [13, 12] = 30.32 , [14, 1] = 16 ,
[14, 2] = 16.24.25.26.27.28.29.30.31.32 , [14, 3] = 18.20.26.27.29.31.32.33 ,
[14, 4] = 20.30.32 , [14, 5] = 24.26 , [14, 6] = 30 , [14, 7] = 24.26.31.32 ,
[14, 8] = 30.31.32.33 , [14, 9] = 28.32 , [14, 10] = 30.33 , [14, 11] = 33 ,
[14, 12] = 31.32 , [15, 1] = 32 , [15, 2] = 17 ,
[15, 3] = 19.20.21.22.23.24.26.33 , [15, 5] = 20.22.23.24.27.28.29.33 ,
[15, 6] = 20.33 , [15, 7] = 22.25.26.31.32 , [15, 8] = 25.26.30.31.33 ,
[15, 9] = 27.30.32 , [15, 10] = 32 , [15, 11] = 30 , [15, 12] = 32 ,
[15, 14] = 33 , [16, 1] = 20.30.32 , [16, 2] = 18 , [16, 3] = 22.26.27.30.31.33 ,
[16, 4] = 22.32 , [16, 5] = 24.26.28.29.32 , [16, 6] = 24.33 ,
[16, 7] = 26.28.29.30.31 , [16, 8] = 28.29.30.32.33 , [16, 9] = 29.30.32 ,
[16, 10] = 32.33 , [16, 11] = 31.32 , [16, 13] = 33 , [17, 1] = 19 ,
[17, 2] = 19.20.21.33 , [17, 3] = 20.21.22.23.24.26 , [17, 4] = 20 ,
[17, 5] = 24.25.26.27.28.29.30.31.32.33 , [17, 6] = 33 ,
[17, 7] = 24.25.26.27.28.32 , [17, 8] = 28.32.33 , [17, 9] = 28.30.31.32 ,
[17, 10] = 33 , [17, 15] = 33 , [18, 1] = 26.27 , [18, 2] = 26.27.30.31.33 ,
[18, 3] = 30.31.32.33 , [18, 4] = 30.33 , [18, 5] = 33 , [18, 7] = 33 ,
[19, 1] = 20 , [19, 2] = 21 , [19, 3] = 22.23.25.26.28.32.33 , [19, 4] = 22 ,
[19, 5] = 24.25.26.27.29.31 , [19, 6] = 24.25.30.33 , [19, 7] = 26.27.28.31.33 ,
[19, 8] = 28.30.31.33 , [19, 9] = 29.32.33 , [19, 10] = 33 , [19, 11] = 33 ,
[19, 12] = 33 , [19, 13] = 33 , [20, 1] = 22 , [20, 2] = 22.24.33 ,
[20, 3] = 24.26 , [20, 5] = 28.31.32.33 , [20, 7] = 32 , [20, 8] = 30.33 ,
[20, 9] = 30.33 , [20, 12] = 33 , [21, 1] = 23 , [21, 2] = 23.25.26.27.29.31.32 ,
[21, 3] = 25.26.28.32 , [21, 4] = 25 , [21, 5] = 28.31.32 , [21, 7] = 28.29.30.31 ,
[21, 8] = 30 , [21, 9] = 31.33 , [21, 10] = 33 , [21, 11] = 33 , [22, 2] = 24 ,
[22, 3] = 26.28.29.30.32 , [22, 5] = 29.30 , [22, 6] = 30.33 , [22, 7] = 30.32 ,
[22, 8] = 30.33 , [22, 9] = 32 , [22, 11] = 33 , [23, 1] = 25 ,
[23, 2] = 26.27.29.31.32.33 , [23, 3] = 27 , [23, 4] = 27 , [23, 5] = 28.29.32 ,
[23, 6] = 28.30.31 , [23, 7] = 29.30.31.32.33 , [23, 8] = 31.32.33 , [24, 1] = 26 ,
[24, 2] = 26.28.30.33 , [24, 3] = 28.29.30.32 , [24, 4] = 30.33 , [24, 5] = 30.31 ,
[24, 7] = 30.32 , [24, 8] = 33 , [25, 1] = 27 , [25, 2] = 27.28.31 ,
[25, 3] = 28.29.31.33 , [25, 5] = 30.31.33 , [25, 7] = 30.32.33 , [25, 8] = 33 ,
[26, 1] = 30.33 , [26, 2] = 28 , [26, 3] = 29.30.31 , [26, 4] = 32 ,
[26, 5] = 30.32.33 , [26, 6] = 30.33 , [26, 7] = 32 , [27, 2] = 28.31 ,
[27, 3] = 29.30.31.32.33 , [27, 5] = 30.32 , [27, 6] = 30 , [27, 7] = 32 ,
[28, 1] = 29 , [28, 2] = 29.30.31 , [28, 3] = 30.31.32 , [28, 4] = 30 ,
[28, 5] = 33 , [28, 7] = 33 , [29, 1] = 30 , [29, 2] = 31 , [29, 3] = 32.33 ,
[29, 4] = 32 , [29, 5] = 33 , [29, 6] = 33 , [30, 1] = 32 , [30, 2] = 32.33 ,
[30, 3] = 33 , [31, 1] = 33 , [31, 2] = 33 , [32, 2] = 33 .

Computer collection of eighth powers of an appropriate set of words reveals that Γ_f does not have exponent 8, but that

$$\Gamma_f^8 = \langle\langle (YZ^4Y^2Z^2)^8 \rangle\rangle_{\Gamma_f} = \langle 32, 33 \rangle ,$$

so $\Gamma^* = \Gamma/\langle\langle (YZ^4Y^2Z^2)^8 \rangle\rangle_\Gamma$. It follows that Γ^* has a maximal nilpotent quotient, Γ_f^*, of order 2^{31} and class 19. (Of course $\Gamma = \Gamma_f$ implies that $\Gamma^* = \Gamma_f^*$.) A consistent power commutator presentation for Γ_f^* can be read from the consistent power commutator presentation given above for Γ_f by setting generators 32 and 33 trivial.

A coset enumeration based proof for the finiteness of Γ is not yet available. Attempts at proofs like those given in Chapter 3 have not succeeded because of an inability to define sufficient cosets. Attempts have been made with presentations for Γ^* and with presentations for low index subgroups of Γ^*, with a limit of 40000 cosets. (It is appropriate to note that the checking of the proof of Chapter 2 done by R.I. Yager involved use of some special purpose programs that he wrote to manipulate words.)

The rank of the multiplicator of Γ is 9 as is the rank of the multiplicator of Γ^*. By investigation of groups with 11 relators selected from the presentations for Γ and Γ^* the following presentations were found.

The presentation

$$\begin{aligned}\langle Y, Z \mid Y^4 = Z^8 = (YZ^{-1})^2 = (YZ)^4 = (Y^2Z^2)^8 = (YZ^2)^8 = (Z^4Y^2)^8 = (Z^3Y)^8 \\ = (Z^3Y^2)^8 = (Z^2Y^2ZY)^8 = (Z^4YZ^2Y^2ZY)^8 = 1\rangle\end{aligned}$$

of a group Γ_m, say, has Γ as its maximal nilpotent quotient and a minimal number of relations. Replacing the relation $(Z^4Y^2)^8 = 1$ by $(YZ^4Y^2Z^2)^8 = 1$ gives a presentation for a group Γ_m^* which has Γ^* as its maximal nilpotent quotient and has a minimal number of defining relations. Questions as to whether $\Gamma_m = \Gamma$ and $\Gamma_m^* = \Gamma^*$ are as yet unsolved.

The fact that a minimal presentation for G was found in Chapter 3 enables us to simplify the presentations for Γ and Γ^* somewhat. A study of the proof that Γ is finite reveals that the relations (ix), (xi), (xxi), (xxii), (xxiii), (xxiv), (xxv) given in the presentation for Γ on p. 124 are used only to prove that the relations (8), (10), (12), (13), (14) and (15) (Chapter 2, Theorem 3) hold in Γ. In view of the minimal presentation for G, this second set of relations is not required to prove the finiteness of G, so it follows that the first set is not required in the presentation for Γ. Further relation (xvi) is redundant because it is used only to prove that relation (27) holds in Γ. This is used only to prove Lemma 5 (19) which is not used elsewhere, so is not essential to the proof.

The above information gives us an 18 relator presentation for Γ (relations (i)-(viii), (x), (xii)-(xv), (xvii)-(xx)) and a 19 relator presentation for Γ^* (namely the 18 relations for Γ plus $(YZ^4Y^2Z^2)^8 = 1$).

Recall that G was selected as a preimage of a subgroup of Γ^* , namely $\langle (ZY)^2Z^{-2}, (YZ)^2Z^{-2} \rangle$, say S . A program auxiliary to the NQA enables computation of a subgroup of a group given generators for the subgroup and a consistent power commutator presentation for the group. The subgroup S in Γ^* was computed and found to have order 2^{14} .

It follows that the mapping of G onto S has kernel of order 2^7 . The availability of the consistent power commutator presentation for Γ^* enables us to compute generators for the kernel of this mapping. By mapping elements of the kernel into B we find words in the generators of B which are trivial in B .

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