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APPLICATION OF COMPUTERS TO QUESTIONS LIKE THOSE OF BURNSIDE

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Computers have been used in seeking answers to questions related to those about periodic groups asked by Burnside in his influential paper of 1902. A survey is given of results obtained with the aid of computers and a key program which manipulates presentations for groups of prime-power order is described.

1. Introduction

Computers are becoming increasingly important in studying questions about groups like those about periodic groups raised by Burnside in his influential paper of 1902. The actual questions asked by Burnside are discussed in the paper Problems in these proceedings. He was concerned with what are now called Burnside groups. We consider here similar questions about a wider class of groups whose description may include individual relations and residual finiteness in addition to an exponent law. For simplicity we refer to all these questions as Burnside questions. The main results are described in section three below.

This report has its origins in a lecture given by one of us at the workshop. Since then both of us have lectured a number of times on this theme. There is a brief report on this subject in the lecture "Computers in Group Theory" given by Marshall

Hall to the Galway Summer School in 1973 (see M. Hall 1977, pp. 34-35).

The growing impact of computers derives from their capacity for fast accurate symbol manipulation on a grand scale. The symbol manipulation procedure relating to groups for which computers have been most extensively used is coset enumeration. A detailed account can be found in a paper by Cannon, Dimino, Havas and Watson (1973). It has been widely used in the context of Burnside questions. For instance, given a finite set $X = \{a, b\}$ and the finite set

$$R = \{a^4, b^4, (ab)^4, (a^{-1}b)^4, (a^2b)^4, (ab^2)^4, (a^2b^2)^4, (a^{-1}b^{-1}ab)^4, (a^{-1}bab)^4\}$$

of group words on X, coset enumeration yields that the group G generated by X and having R as a defining set of relators has order $4096 = 2^{12}$ and a pair of permutations on 4096 letters which generate a group isomorphic to G. On a Univac 1100/42 using the coset enumeration program, Todd-Coxeter V2.2, this result can be obtained in about 30 seconds. From the output it can be deduced that G has exponent 4. Hence $B(2, 4)$ has order 2^{12} and has a presentation on two generators which has a defining set of only nine fourth powers. This example shows, in a simplified form, the role of coset enumeration in the study of groups. This is to find, where possible, the set $G\backslash S$ of cosets of a subgroup S in a group G and a representation of G by permutations of $G\backslash S$ given a finite presentation for G and a finite generating set for S. Further examples of results obtained using coset enumeration are given in section three.

In the context of the Burnside questions good progress has been made using computers to perform manipulations with descriptions of groups of prime-power order. For example, $B(2, 4)$ cannot be defined by fewer than nine fourth powers (Macdonald 1974), $\overline{B}(2, 5)$, the largest residually finite quotient of $B(2, 5)$, has order 5^{34} (Havas, Wall and Wamsley 1974), $B(4, 4)$ has order 2^{422} and $\overline{\Delta}$, the largest residually finite quotient of

$$\Delta := \langle x, y \mid x^2 = y^4 = 1, \text{ exponent } 8 \rangle ,$$

has order dividing 2^{205}. In section three we will give an outline of how the latter results were obtained. Recently the same method has been used to obtain a related result; see the note by Alford and Pietsch in these proceedings. The programs used are based on algorithms for calculating information about nilpotent quotients of groups given by finite presentations and finite sets of identical relations or laws. There have been a few rather brief accounts of such nilpotent quotient programs published (Macdonald 1973, 1974, Wamsley 1974, Newman 1976). While these describe applications to Burnside questions, none of them goes into sufficient detail to discuss a key point in such applications which is how to make law enforcement practical. This point is taken up in section two of this account.

Computers have also been programmed to perform other symbol manipulation procedures related to groups. Two such procedures which are playing an important role in some current investigations are the Reidemeister-Schreier algorithm for calculating presentations for subgroups of finite index in finitely presented groups (Havas 1974) and manipulations such as Nielsen and Tietze transformations with presentations.

The programs we use are written in a readily portable superset of 1966 Standard FORTRAN. Further information on both programs and results can be obtained from the authors.

We are indebted to many colleagues, too many to name individually, for helpful discussions and access to programs. We thank them all collectively.

2. A nilpotent quotient program

In this section we describe a computer program, the Canberra nilpotent quotient program, which, with its forerunners, played an important role in many of the applications to be discussed in the next section. The program provides the capacity to manipulate presentations for groups of prime-power order efficiently. This makes it possible to study many groups of order p^n for moderate values of p and n, say $p < 1000$, $n < 20$, and selected groups with n well into the hundreds and occasionally even higher. There have been essentially two previous programs of this kind; one produced by Macdonald (1974) and one by Bayes, Kautsky and Wamsley (1974). The Canberra program has its origins in the latter and incorporates some important features from it.

The core of the program is a set of routines for manipulating what we call power-commutator presentations. These are finite presentations which present groups of prime-power order. Moreover every group of prime-power order has such presentations. A *power-commutator presentation* consists of a finite set $A = \{a_1, \ldots, a_n\}$ of generators and a finite set of $n(n+1)/2$ relations

$$a_i^p = \prod_{l=i+1}^{n} a_l^{\alpha(i,l)} ,$$

$$a_j a_i = a_i a_j \prod_{l=j+1}^{n} a_l^{\alpha(i,j,l)} , \quad i < j , \tag{2.1}$$

where p is a prime and $\alpha(i, l)$, $\alpha(i, j, l)$ belong to $\{0, 1, \ldots, p-1\}$. It will be convenient to denote this presentation $(n; p, \alpha)$ and the group it presents $\langle n; p, \alpha\rangle$. The way in which such presentations may be thought of as occurring will be explained later. The word "commutator" is used because in most contexts the second type of relation is written with a commutator on the left; it is convenient here to defer this change a little.

It is easy to see that every element of $\langle n; p, \alpha\rangle$ can be represented by a word

of the form $\prod_{i=1}^{n} a_i^{\xi(i)}$ with $\xi(i)$ in $\{0, 1, \ldots, p-1\}$; we call such a word *normal* and abbreviate it to ξ , and use $\emptyset$ for the identity element. At the heart of this whole approach is the simple fact that the given set of relations makes it straightforward to calculate a normal word equivalent to the product of two normal words. It will be convenient to discuss this (collection) process in the wider context of all semigroup words on A ; inverses play little role in the present discussion. Notice that the set of left sides of the relations in $(n; p, \alpha)$ is precisely the set of minimal non-normal semigroup words on A . A *collection process* relative to $(n; p, \alpha)$ is a procedure of the following kind:

(2.2) Given a semigroup word w on A ,

1 if w is normal, stop;

2 replace a minimal non-normal subword u of w by the right side of the relation whose left side is u , and return to 1 .

It is routine to prove that collection processes terminate, in a normal word. There are many collection processes because of the choice in step 2. A detailed discussion of collection in the Canberra program is given in a paper by Havas and Nicholson (1976). The rest of this account is based on the collection process in the Canberra program; it would apply equally well to any other (fixed) collection process. The normal word resulting from applying this collection to a word w will be denoted (w) .

The first question which arises is: what is the order of $\langle n; p, \alpha \rangle$? Since the number of normal words is p^n , the group $\langle n; p, \alpha \rangle$ has order at most p^n . Unfortunately it often happens that $\langle n; p, \alpha \rangle$ has order less than p^n . In this case it is easy to see that there are normal words ξ, η, ζ such that $((\xi\eta)\zeta) \neq (\xi(\eta\zeta))$. This leads to the following straight-forward test for determining whether the order of $\langle n; p, \alpha \rangle$ is p^n . Let T_n be the set of words consisting of all

$$a_k a_j a_i , \qquad i < j < k ,$$

$$a_j^p a_i , \; a_j a_i^p , \qquad i < j ,$$

$$a_i^{p+1} .$$

Observe that each word in T_n has exactly two minimal non-normal subwords and that these overlap. Hence each word in T_n can be uniquely written $\xi\eta\zeta$ with ξ, η, ζ non-empty normal words and $\xi\eta, \eta\zeta$ minimal non-normal. The group $\langle n; p, \alpha \rangle$ has order p^n if and only if for each word $\xi\eta\zeta$ in T_n , $((\xi\eta)\zeta) = (\xi(\eta\zeta))$. If this

is satisfied $(n, p; \alpha)$ is said to be *consistent*. If $(n; p, \alpha)$ is not consistent, then some test word will collect to two different normal words.

This leads to another basic procedure, reduction, which, given a power-commutator presentation $(n; p, \alpha)$ and a pair μ, ν of distinct normal words, produces on a subset of (n-1 elements of) A a power-commutator presentation for the group defined by $(n; p, \alpha)$ and $\mu = \nu$. It is not difficult to describe reduction in general; however, it is easier to describe a special case which suffices for the present discussion. Before doing so, we complete our consideration of the order question. When the words μ, ν come from collecting a word in T_n the group defined by the resulting power-commutator presentation is clearly isomorphic to $\langle n; p, \alpha \rangle$. Repeating this process leads, eventually, to a consistent power-commutator presentation for $\langle n; p, \alpha \rangle$; the order of $\langle n; p, \alpha \rangle$ can be read off from this.

From now on the relations in a power-commutator presentation will be taken in power and commutator form. Let us say that a normal word ξ is *obviously central and of order* p if, for every generator a_l which occurs in ξ (that is, $\xi(l) \neq 0$), all the relations with a_l on the left have $\emptyset$ on the right. We only need a *reduction* procedure when the words μ, ν are obviously central and of order p. Let G be the group defined by $(n; p, \alpha)$ and $\mu = \nu$. Since μ, ν are distinct there is a greatest positive integer k such that $\mu(k) \neq \nu(k)$. Let $\eta(l)$ be the least non-negative integer such that

$$\eta(l)\big(\mu(k)-\nu(k)\big) \equiv \nu(l) - \mu(l) \text{ modulo } p .$$

The relation $\mu = \nu$ is equivalent to

$$a_k = \prod_{l=1}^{k-1} a_l^{\eta(l)} \tag{2.3}$$

(with a_l obviously central and of order p whenever $\eta(l) \neq 0$). Let α' be the function defined by

$$\alpha'(i, l) \equiv \alpha(i, l) + \eta(l)\alpha(i, k) \text{ modulo } p ,$$

$$\alpha'(i, j, l) \equiv \alpha(i, j, l) + \eta(l)\alpha(i, j, k) \text{ modulo } p ,$$

and

$$0 \leq \alpha'(i, l) , \quad \alpha'(i, j, l) < p .$$

Then G is defined by $(n; p, \alpha')$ and 2.3. Since $\alpha'(i, k) = \alpha'(i, j, k) = 0$ for all i, j, the generator a_k only occurs on the left of relations in $(n; p, \alpha')$; and when it does the relation is a consequence of 2.3 and relations not involving a_k. Hence the presentation obtained from $(n; p, \alpha')$ by deleting all relations involving a_k and a_k from the generating set is a power-commutator presentation for G on $n - 1$ generators; this is what was required.

Clearly it would be desirable from the point of view of computational efficiency to make the set of test words for consistency smaller. There are some obvious redundancies; for example, one never needs to test $a_n a_j a_i$. We now introduce some further structure into power-commutator presentations which leads to a useful reduction in the test set for consistency. This additional structure also leads to a criterion which makes it practical to test whether a group given by a power-commutator presentation has exponent p^f.

(2.4) A power-commutator presentation will be called *weighted* if

(i) some of its relations are called *definitions*; and

(ii) it admits a weight function ω from $\{1, \ldots, n\}$ to the positive integers such that

(a) $\omega(1) = 1$ and $\omega(i) \leq \omega(i+1)$;

(b) for each k with $\omega(k) > 1$ there is exactly one definition whose right side is a_k;

(c) if the definition of a_k is $a_i^p = a_k$, then $\omega(k) = \omega(i) + 1$;

(d) if the definition of a_k is $[a_j, a_i] = a_k$, then $\omega(k) = \omega(i) + \omega(j)$;

(e) if $\alpha(i, l) \neq 0$, then $\omega(l) \geq \omega(i) + 1$,
if $\alpha(i, j, l) \neq 0$, then $\omega(l) \geq \omega(i) + \omega(j)$.

Note there may be more than one relation which has a_k as right side but only one of these is the definition of a_k. It is easy to see that there are power-commutator presentations which admit no such weight function; indeed presentations of this kind occur in the course of the calculations to be described. A weighted power-commutator presentation $(n; p, \alpha)$ with weight ω will be denoted $(n; p, \alpha; \omega)$. The positive integer $\omega(n)$ is the *class* of $(n; p, \alpha; \omega)$ and of $\langle n; p, \alpha; \omega \rangle$, the group it defines. This class is, see 2.6, the exponent-p-central class which is usually defined in terms of the lower exponent-p-central series of a group G; that is, the chain

$$G = G_0 \geq \ldots \geq G_h \geq G_{h+1} \geq \ldots$$

of subgroups where $G_{h+1} = [G_h, G]G_h^p$. If $G_{c-1} > G_c = E$, then G has *exponent-p-central* class c. Note that a group with exponent-p-central class c is nilpotent and has nilpotency class at most c. In the study of groups of prime-power order the exponent-p-central class often seems the more significant. An important observation about the lower exponent-p-central series is

(2.5) $$[G_j, G_h] \leq G_{j+h+1} .$$

This is proved in essentially the same way as the corresponding result for the lower central series.

(2.6) The exponent-p-central class of $\langle n; p, \alpha; \omega \rangle$ is $\omega(n)$.

Let $G = \langle n; p, \alpha; \omega \rangle$. We prove that G_h is generated by all the a_k with $\omega(k) \geq h + 1$. This is clear for $h = 0$. For $h \geq 1$, suppose G_{h-1} is generated by all the a_h with $\omega(l) \geq h$. It follows that G_h is generated by all the a_l^p and all the $[a_l, a_i]$ with $\omega(l) \geq h$ and a_i arbitrary, and so, by clause (a) of the definition of a weighted power-commutator presentation, G_h is contained in the subgroup generated by all a_k with $\omega(k) \geq h + 1$. Conversely, if $\omega(k) \geq h + 1 > 1$, then either $a_k = a_i^p$ with $\omega(k) = \omega(i) + 1$ or $a_k = [a_j, a_i]$ with $\omega(k) = \omega(i) + \omega(j)$. Hence, by the inductive assumption and 2.5, a_k belongs to G_h .

A first gain which comes from using weighted power-commutator presentations is that the set T_n of test words for consistency can be reduced to T_n^* :

(2.7) T_n^* is the set of all words

$$a_k a_j a_i , \qquad i < j < k ,$$

with $\omega(i) + \omega(j) + \omega(k) \leq \omega(n)$,

$$a_j^p a_i , \; a_j a_i^p , \quad i < j ,$$

with $1 + \omega(i) + \omega(j) \leq \omega(n)$, and

$$a_i^{p+1} , \qquad \text{with } 1 + 2\omega(i) \leq \omega(n) .$$

This is so because it is straight-forward to check that for every other test word collection yields the same result; for example, if $1 + 2\omega(i) > \omega(n)$, then $\left(a_i^p\right)a_i = a_i\left(a_i^p\right)$ since $\left(a_i^p\right)$ only involves a_l with $\omega(l) \geq \omega(i) + 1$. While this is a useful reduction, in practice redundancy is usually observed once the class exceeds 3 ; moreover this redundancy increases with class. It would be desirable to have this observation reflected in still smaller test sets.

A second gain is a criterion which makes testing whether an exponent law holds reasonably practical. For a normal word ξ , let us call $\sum_{l=1}^{n} \omega(l)\xi(l)$ the *weight* of ξ .

(2.8) The group $\langle n; p, \alpha; \omega \rangle$ has exponent p^f provided $\xi^{p^f} = \emptyset$ for all normal words of weight at most $\omega(n)$.

The criterion can be proved by a simple varietal argument. The crux is covered by an argument of Higman (1959, p. 169; or see §3.3 of Hanna Neumann's book (1967) where, however, it is not as conveniently put). It is applied to the word $(x_1 \ldots x_s)^{p^f}$ in the free group F freely generated by $X = \{x_1, \ldots, x_s\}$ to give

$$(x_1 \ldots x_s)^{p^f} = \prod_J \left(x_J^{p^f}\right)^{\pm 1} . u$$

where x_J is the product of some proper (ordered) subset of X and u is a product of commutators in X each of which involves each element of X . From this the criterion is proved by induction. Let ξ be a normal word. If the weight of ξ is at most $\omega(n)$, then clearly $\xi^{p^f} = \emptyset$. Otherwise assume $\eta^{p^f} = \emptyset$ whenever the weight of η is less than that of ξ . Let $s = \sum_{l=1}^{n} \xi(l)$. Let θ be the homomorphism from F to $\langle n; p, \alpha; \omega \rangle$ which maps x_k to a_i where

$$\sum_{l=1}^{i-1} \xi(l) < k \leq \sum_{l=1}^{i} \xi(l) ,$$

then $(x_1 \ldots x_s)\theta = \xi$. Hence by the above

$$\xi^{p^f} = \prod_J \left((x_J\theta)^{p^f}\right)^{\pm 1} . u\theta .$$

Clearly $x_J\theta$ is a normal word and its weight is less than that of ξ , so $(x_J\theta)^{p^f} = \emptyset$. From the structure of u it follows that $u\theta$ is a product of commutators in A each with at least $\xi(l)$ entries a_l for each l . Since the weight of ξ exceeds $\omega(n)$ and $\langle n; p, \alpha; \omega \rangle$ has class $\omega(n)$, it follows readily that $u\theta = \emptyset$. Hence $\xi^{p^f} = \emptyset$ as required.

This criterion is incorporated into the program of Bayes, Kautsky and Wamsley (though their paper (1974) does not mention it). Macdonald's (1974) criterion seems to be of a similar kind. While the criterion makes testing whether exponent laws hold quite practical, the calculation of even the powers specified by it is the most severe limitation to these explorations of groups of Burnside type. As with consistency testing much redundancy is observed, so it would be desirable to find further restrictions on the powers that need to be calculated.

In practice we have used other considerations to extend the scope of what can be done. In our initial work on $B(4, 4)$ we used related commutator laws and congruences. The relevant calculations are more complicated to program, but are then quicker to execute. The simplest example of this comes from the law

$$[y, x]^2[y, x, x, x][y, x, y, x][y, x, y, y] = \emptyset$$

which holds in $B(2, 4)$ and therefore in all groups of exponent 4. This yields the criterion that $\langle n; 2, \alpha; \omega\rangle$ has exponent 4 provided $a_k^4 = \emptyset$ for all k and

$$[\eta, \xi]^2[\eta, \xi, \xi, \xi][\eta, \xi, \eta, \xi][\eta, \xi, \eta, \eta] = \emptyset$$

for all normal words ξ, η such that $\xi\eta$ is a normal word of weight at most $\omega(n)$. More complicated criteria come from allowing more variables. The present nilpotent quotient program can handle $B(4, 4)$ without such special considerations. We hope, if resources permit, to implement these more refined methods in such a way that they will allow the routine testing of other laws such as commutator laws. In studying $\overline{\Delta}$ our main goal was to prove finiteness. For this it suffices to show that some preimage of $\overline{\Delta}$ is finite; or, equivalently, to find a set E of eighth powers such that the residually finite group generated by $\{x, y\}$ with $\{x^2, y^4\} \cup E$ as a defining set of relators is finite. Therefore we used heuristic considerations to limit the eighth power calculations done; more details are given in the next section.

Just as reduction is used to make a power-commutator presentation consistent, so it can be used to enforce exponent laws; that is, given p^f, to yield from $(n; p, \alpha)$ a power-commutator presentation for the largest exponent p^f quotient of $\langle n; p, \alpha\rangle$. If testing shows that $\xi^{p^f} \neq \emptyset$ for some ξ, then the largest exponent p^f quotient of $\langle n; p, \alpha\rangle$ is the same as that of the group defined by $(n; p, \alpha)$ and $\xi^{p^f} = \emptyset$. Applying reduction yields a power-commutator presentation for this group which has fewer generators. Repeating this process of testing and reduction clearly leads to the required power-commutator presentation.

We introduce next the p-covering presentation for a consistent weighted power-commutator presentation $(n; p, \alpha; \omega)$. Let $P = \langle n; p, \alpha; \omega\rangle$. Suppose $\omega(d) = 1 < \omega(d+1)$, then P can be generated by d, and no fewer, elements. The *p-covering presentation* for $(n; p, \alpha; \omega)$ has a generating set $a_1, \ldots, a_{d+n(n+1)/2}$ and relations of three kinds:

(1) the $n - d$ definitions in $(n; p, \alpha; \omega)$,

(2) $d + n(n-1)/2$ of the form $u_k = v_k a_k$ for $k \in \{n+1, \ldots, d+n(n+1)/2\}$ where $u_k = v_k$ is a relation in $(n; p, \alpha; \omega)$ which is not a

definition,

(3) those which specify that $a_{n+1}, \ldots, a_{d+n(n+1)/2}$ are central and of order p.

Relations of the first two kinds are definitions. Because $(n; p, \alpha; \omega)$ is consistent, if the consistency test fails for a test word relative to the p-covering presentation, the resulting normal words are obviously central and of order p, and so reduction can be used to yield a consistent power-commutator presentation $(n^*; p, \alpha^*)$ with $n^* - d$ definitions. This presentation can not in general be made into a weighted presentation. The group $P^* = \langle n^*; p, \alpha^* \rangle$ is the *p-covering group* of P. Clearly P^* has class at most $\omega(n) + 1$. If Q is a group of class $\omega(n) + 1$ whose largest class $\omega(n)$ quotient is P, it can be shown that Q is a homomorphic image of P^*. If the class of P^* is $\omega(n)$, there is no such Q and P is said to be *terminal*; otherwise, there is such a Q and P is said to be *capable*. If the relations $u_k = v_k$ which are not definitions are suitably ordered then the class of P^* can be determined from the definition of a_{n+1} in $(n^*; p, \alpha^*)$. If this is $a_i^p = a_{n+1}$ with $\omega(i) = \omega(n)$ or $[a_j, a_i] = a_{n+1}$ with $\omega(i) + \omega(j) = \omega(n) + 1$, then the class of P^* is $\omega(n) + 1$ and P is capable; otherwise the class is $\omega(n)$ and P is terminal. The kernel of the mapping from P^* to P is isomorphic to the second homology group $H_2(P, Z_p)$ of P over the field of p elements; we call it the *p-multiplicator* of P. The rank $n^* - n$ of the p-multiplicator is the least number of relations needed to define P as a group of p-power order on d generators; in saying a presentation $\{X|R\}$ defines a group G as a group of p-power order, we mean that the group $\langle X|R \rangle$ has a largest quotient of p-power order and that this quotient is isomorphic to G. It follows that $n^* - n$ is a lower bound on the number of relations needed to define P as a d generator group. It remains an open question whether this lower bound is always attained.

Since the class of P^* is at most $\omega(n) + 1$, it is possible to introduce fewer generators in writing down a power-commutator presentation for P^* unless $\omega(n) = 1$. It suffices to add a new generator for each relation in $(n; p, \alpha; \omega)$ which neither is a definition nor has its left side of the form $[a_j, a_i]$ with $\omega(i) + \omega(j) \geq \omega(n) + 2$. This is of value because it means fewer reduction steps are needed to reach a consistent power-commutator presentation. Further cuts are possible. For example, if $\omega(k) = 2$ and $[a_j, a_i] = a_k$ is the definition of a_k and if $\omega(l) = \omega(n) - 1$, then

$$[a_l, a_k] = [a_l, a_j, a_i][a_l, a_i, a_j]^{p-1} .$$

To tell much more of this story requires technical details that it is not appropriate to go into here. Note though that it implies that $\omega(i) = 1$ in all definitions of

the form $[a_j, a_i] = a_k$.

We have now developed enough machinery to outline how to determine a consistent power-commutator presentation for the largest class c quotient of the Burnside group $B(d, p^f)$ which we denote $B(d, p^f; c)$. Begin with a consistent weighted power-commutator presentation for $B(d, p^f; 1)$: it has a generating set $\{a_1, \ldots, a_d\}$ with relations $a_i^p = \emptyset$ and $[a_j, a_i] = \emptyset$ for all i, j and weight function the constant function with value 1 . For $c > 1$, let $(n; p, \alpha; \omega)$ be a consistent weighted power-commutator presentation for $B(d, p^f; c-1)$ with $\omega(n) = c - 1$. Write down a power-commutator presentation for the p-covering group of $\langle n; p, \alpha; \omega \rangle$ and reduce it to a consistent presentation $(n^*; p, \alpha^*)$. If this shows $\langle n; p, \alpha; \omega \rangle$ is terminal, then the largest residually finite quotient, $\overline{B}(d, p^f)$, of $B(d, p^f)$ is presented by $(n; p, \alpha; \omega)$. It follows that $\overline{B}(d, p^f)$ has order p^n , class $c - 1$ and can be defined as a group of p-power order by d generators and $n^* - n$ p^fth powers. If $\langle n; p, \alpha; \omega \rangle$ is capable, enforce the exponent p^f on $(n^*; p, \alpha^*)$; note that in this context the normal word collected from a p^fth power will be obviously central and of order p , so the condition needed for reduction holds. Let the resulting (consistent) presentation be $(n'; p, \alpha')$. If $n' = n$, then again $\overline{B}(d, p^f)$ is presented by $(n; p, \alpha; \omega)$ and the other consequences follow. For example, a power-commutator presentation for $B(2, 4; 5)$ has 12 generators, and that of its 2-covering group has 21 generators; this latter presentation shows $B(2, 4; 5)$ is capable, however enforcing exponent 4 reduces it to the presentation for $B(2, 4; 5)$; hence $\overline{B}(2, 4)$ has order 2^{12} , class 5 and needs 9 relations, which can be 4th powers, to define it as a group of 2-power order; since $B(2, 4)$ is well-known to be finite, these conclusions apply to it as well. The function ω can be extended to a weight function ω' on $(n'; p, \alpha')$ by: $\omega'(i) = \omega(i)$ for $i \leq n$ and $\omega'(i) = c$ for $i > n$. Thus this procedure can be iterated. If $\overline{B}(d, p^f)$ is finite, as it is known to be when $f = 1$ by a result of Kostrikin, the procedure will eventually stop. In practice (for $p \geq 5$) this has happened so far only for $\overline{B}(2, 5)$.

In the study of groups, such as Δ , whose description includes individual relations, we work with presentations which, in effect, combine the given finite presentation and an appropriate power-commutator presentation. Thus in studying a group given by the finite presentation

$$\{b_1, \ldots, b_m; u_1 = v_1, \ldots, u_r = v_r\} ,$$

we work with augmented power-commutator presentations which have

$\{b_1, \ldots, b_m, a_1, \ldots, a_n\}$ as set of generators, and relations

$$u_1 = v_1, \ldots, u_r = v_r, \quad b_h = \prod_{l=1}^{n} a_l^{\rho(h,l)}, \quad 1 \le h \le m,$$

with $\rho(h, l)$ in $\{0, 1, \ldots, p-1\}$, and all the relations 2.1.

Where relevant, there is a weight function ω as before on $\{1, \ldots, n\}$; however, now there is a definition for each a_k, the first d definitions, where $\omega(d) = 1 < \omega(d+1)$, have left side b_h for some h.

In this context every semigroup word on the generators can be collected to a normal word on $\{a_1, \ldots, a_n\}$ by adding the steps: replace b_h by $\prod a_l^{\rho(h,l)}$. Where the inverse of a generator b_i is involved we deal with it by introducing a new generator b_{m+i} and including among the defining relations $b_i b_{m+i} = \emptyset$. We say the presentation is *consistent* if, in addition to the previous conditions, it satisfies $(u_k) = (v_k)$, where $(\ )$ denotes the result of collection, for all k in $\{1, \ldots, r\}$. Thus, if an augmented power-commutator presentation is consistent, the group it defines has order p^n and is a quotient of the group given by the input presentation. We next modify the procedure for writing down p-covering presentations. A new generator is added to each relation which is neither given nor a definition (nor has large weight left side). Note this includes the case when $n = 0$; then there are no definitions and the p-covering presentation has generating set $\{b_1, \ldots, b_m, a_1, \ldots, a_m\}$ and relations

$$u_1 = v_1, \ldots, u_r = v_r,$$

$$b_k = a_k, \quad 1 \le k \le m,$$

and each a_i is central and has order p.

We can finally outline how to determine a consistent augmented power-commutator presentation for the largest class c quotient of

$$\langle b_1, \ldots, b_m \mid u_1 = v_1, \ldots, u_r = v_r, \text{ exponent } p^f \rangle .$$

Begin with the largest class 0 quotient: it has generating set $\{b_1, \ldots, b_m\}$ and relations

$$u_1 = v_1, \ldots, u_r = v_r, \quad b_1 = \emptyset, \ldots, b_m = \emptyset .$$

For $c \ge 1$, let there be given a consistent augmented power-commutator presentation $\mathcal{P}$ for the largest class $c - 1$ quotient. Write down the p-covering presentation of it and reduce to a consistent presentation. If this is $\mathcal{P}$, the given group has a largest quotient of p-power order which is presented by $\mathcal{P}$. Otherwise enforce the

exponent p^f . The same conclusion follows if this reduces the presentation to P . If not, extend the weight function as before.

3. Results

3.1 EXPONENT FOUR

Leech (1963) was the first to publish results on Burnside groups obtained with the aid of a computer. He produced presentations for some groups of exponent three and four. For example he found, using computer coset enumeration, various sets of nine fourth powers which suffice as relators for $B(2, 4)$. He also reported on a group of Burnside type suggested by Philip Hall, namely

$$I3 = \langle a, b, c \mid a^2 = b^2 = c^2 = \emptyset, \text{ exponent } 4 \rangle .$$

Leech computed the order of $I3$ and found ten relator presentations for this group. With a view towards $B(3, 4)$ he investigated the group

$$I2 = \langle a, b, c \mid a^2 = b^2 = \emptyset, \text{ exponent } 4 \rangle .$$

Macdonald (1973, 1974) used a nilpotent quotient program to prove various results about Burnside groups. He showed that minimal presentations for $B(2, 4)$ have nine relations and for $I3$ have ten relations, as achieved by Leech. He also showed that $I2$ has order 2^{19} . Bayes, Kautsky and Wamsley (1974) computed the order of $B(3, 4)$, 2^{69} , and Alford, Havas and Newman (1975) computed the order of $B(4, 4)$, 2^{422} , in both cases using a nilpotent quotient program. More recently Havas (to appear) has used coset enumerations in $B(2, 4)$ to express the fifth Engel word as a product of fourth powers.

We present here some new results about various groups of exponent four and in particular about $I2$ and about $B(4, 4)$. We include a description of the computations involved.

Using the Canberra nilpotent quotient program we readily confirm Macdonald's result that $I2$ has order 2^{19} . We determine that the rank of its 2-multiplicator is 20 , which shows that at least 18 fourth power relations in addition to the 2 involutory relations are required to provide a presentation for $I2$. The Canberra nilpotent quotient program also produces a list of 18 fourth powers which (together with the two involutory relations) is sufficient to define $I2$ as a group of 2-power order. This list is routinely obtained as the lexicographically shortest words in terms of the power-commutator presentation generators which suffice. This kind of list is quite suitable for nilpotent quotient calculations but less satisfactory for other purposes, such as coset enumeration, because the free group word length is not usually particularly short. In the case of $I2$ one of the longest such words is

$(acac^{-1}acbc^{-1}acab)^4$, a free group word of length 12 , raised to the fourth power. However other sets of fourth powers can be found readily. One approach is to calculate the 2-covering group of $I2$ and then to collect fourth powers in that covering group. Collection of short powers (short in the free group sense) reveals that the following relations suffice to define $I2$ as a group of 2-power order:

$$a^2 = b^2 = c^4 = (ab)^4 = (ac)^4 = (bc)^4 = (abc)^4 = (acb)^4 = (acc)^4 = (bcc)^4$$
$$= (abac)^4 = (abcb)^4 = (abcc)^4 = (acbc)^4 = (acbc^{-1})^4 = (ababc)^4 = (abcbc)^4$$
$$= (bacac)^4 = (bc^{-1}aca)^4 = (bcacc)^4 = \emptyset .$$

Such short relations are much more suitable for coset enumeration. It is easy to enumerate the 8192 cosets of $\langle a, c\rangle$ in the group presented this way. (The corresponding coset enumeration in the group presented with fourth powers routinely given by the Canberra nilpotent quotient program has not been successfully performed.) Since $\langle a, c \mid a^2 = c^4 = (ac)^4 = (acc)^4 = \emptyset\rangle$ presents a group of order 64 (very easy by coset enumeration) it follows that the above 20 relations provide a minimal presentation for $I2$.

Note that some of the claims in the literature about $I2$ are wrong. Thus Leech (1963) gives the incorrect order for the group. This arises from the misapprehension that $\langle ab, c\rangle$ is isomorphic to $B(2, 4)$ of order 2^{12} . In fact $\langle ab, c\rangle$ is a group of order 2^{10} , which follows from the above calculations. This may also be shown by explicit calculation with the power-commutator presentation for $I2$. E. Oppelt of RWTH Aachen has implemented routines which augment the Canberra nilpotent quotient program and provide facilities for subgroup construction and investigation by computer, and which can be used for this purpose.

Leech (1967) endeavours to improve on the presentation for $I2$ and also refers to an attempt by Sinkov at obtaining a presentation for that group (which is the presentation given by Cannon (1974)). The presentations of Leech and Sinkov define different groups of order 2^{21} which have $I2$ as a quotient.

Using the Canberra nilpotent quotient program we have computed the rank of the 2-multiplicator of $B(3, 4)$. This rank is 105 and we have found a set of 105 fourth powers which suffice to define $B(3, 4)$ as a group of 2-power order.

A very significant computational effort was put into determining a consistent power-commutator presentation for $B(4, 4)$. The initial motivation was to gain information on the solubility question for groups of exponent four. At the time that the computations were first performed it seemed possible that the determination of $B(4, 4)$ could perhaps be combined with results of Gupta and Newman (1974) to prove the solubility of all groups of exponent four, if that were in fact the case. Not surprisingly this did not succeed because insoluble groups of exponent four do exist,

as was proved by Razmyslov (1978). However Vaughan-Lee (1979) has used the detailed information available on $B(4, 4)$, via the consistent power-commutator presentation computed for it, as a basis for determining the solubility length of $B(d, 4)$ explicitly in terms of d. As mentioned in section two, a number of technical innovations were made for the initial computation of $B(4, 4)$. A significant saving in computer time was made by the use of formulas instead of direct collection. Whereas the Canberra nilpotent quotient program uses direct collection both to process the consistency equations and to perform exponent testing, in the initial run for $B(4, 4)$ commutator formulas were calculated and used instead, where possible. Thus formulas for Jacobi identities, which correspond to consistency equations, and for fourth power expansions were computed. Substitution in this type of formula is superior to direct collection because the bulk of collection required has been done once and for all in computation of the formula. With improvements in computer technology and the development of the latest version of the Canberra nilpotent quotient program, formula substitution methods are no longer needed in order to compute $B(4, 4)$. Indeed we have used the current version of the Canberra nilpotent quotient program to recompute $B(4, 4)$ and also to compute the rank of its 2-multiplicator, namely 1055. This calculation took about 10 hours on a Univac 1100/42. Note that the rank of the 2-multiplicator shows that a presentation for $B(4, 4)$ requires at least 1055 relations. We have computed a set of 1055 fourth power relations which suffice to define $B(4, 4)$ as a group of 2-power order.

Other groups of exponent four for which we have computed consistent power-commutator presentations are:

$\langle a, b, c \mid a^2 = \emptyset,$ exponent $4\rangle$, which has order 2^{37} and class 7;

$\langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = \emptyset,$ exponent $4\rangle$, which has order 2^{38} and class 5;

$\langle a, b, c, d \mid a^2 = b^2 = c^2 = \emptyset,$ exponent $4\rangle$, which has order 2^{66} and class 6;

$\langle a, b, c, d \mid a^2 = b^2 = \emptyset,$ exponent $4\rangle$, which has order 2^{120} and class 8;

$\langle a, b, c, d \mid a^2 = \emptyset,$ exponent $4\rangle$, which has order 2^{224} and class 10;

$\langle a, b, c, d, e \mid a^2 = b^2 = c^2 = d^2 = e^2 = \emptyset,$ exponent $4\rangle$, which has order 2^{138} and class 6;

$\langle a, b, c, d, e \mid a^2 = b^2 = c^2 = d^2 = \emptyset,$ exponent $4\rangle$, which has order 2^{228} and class 7.

3.2 EXPONENT FIVE

The order of $\overline{B}(2, 5)$ was shown by Havas, Wall and Wamsley (1974) to be 5^{34} using computer calculations in two different ways, one via nilpotent quotient computations and the other using Lie algebraic methods. More recently we have obtained a consistent power-commutator presentation for $B(3, 5; 9)$ using the Canberra nilpotent quotient program. $B(3, 5; 9)$ has order 5^{916} which is less than the order of the free 3-generator Lie algebra satisfying the 4th Engel condition, namely 5^{926}. This provides an independent confirmation of a result of Wall (1974).

Currently a number of people, including M. Hall Jr., G. Havas, J.S. Richardson and J. Wilkinson, are investigating the unrestricted problem for exponent five with computer assistance. Calculations in search of a finiteness proof for $B(2, 5)$ are being performed using nilpotent quotient, coset enumeration, subgroup presentation and presentation manipulation programs.

3.3 EXPONENT SEVEN

We have constructed a consistent power-commutator presentation for $B(2, 7; 13)$. This is a group of order 7^{668}. If x and y are generators of this group then $[y, {}_8x] \neq \emptyset$, so the 8th Engel identity does not hold.

3.4 EXPONENT EIGHT

Detailed results about a number of groups of exponent eight are presented in other papers in these proceedings. Here we present some results and computational details about one particular group of exponent 8.

In the paper by Grunewald, Havas, Mennicke and Newman in these proceedings the problem as to whether the subgroup $\langle a^4, b^2 \rangle$ of $B(2, 8)$ is finite is posed. We show that it has a largest finite quotient by showing that a residually finite preimage is finite.

Let $\Delta := \langle a, b \mid a^2 = b^4 = \emptyset, \text{ exponent } 8 \rangle$. Then Δ has a preimage D which has a maximal nilpotent quotient of order 2^{205} and class 26. This preimage was found by using the exponent testing procedure to partially enforce the exponent law. The class 10 quotient of Δ was calculated in the normal way. With increasing class the exponent tests were taking more and more time. As pointed out in section two, the exponent law test has a lot of redundancy so that, for example, to compute the class 10 quotient of Δ only two eighth powers are actually required, eighth powers of ab and abb which are words of weight 2 and 3. This suggests that it may suffice to test only words of low weight. From class 10 onwards eighth power testing was performed only for words with weight less than 11. D is the group defined by two generators, one of order 2 and one of order 4, in which all normal

words of weight less than 11 have order dividing 8 . To define this as a group of 2-power order 69 eighth powers are needed, with those of highest weight having weight 10 . Subsequently further exponent 8 checking was applied to the consistent power-commutator presentation for the nilpotent quotient of D . It was found that all normal words of weight less than 18 have order dividing 8 . This strongly suggests that the nilpotent quotients of Δ and D are in fact the same. The rank of the 2-multiplicator of the maximal nilpotent quotient of D is 69 .

3.5 EXPONENT NINE

We have computed a consistent power-commutator presentation for $B(2, 9; 12)$ which is a group of order 3^{724} .

Macdonald (1973) points out that "even groups generated by two elements of order 3 lead to depressingly large results", and this is confirmed by Shield (1977). Macdonald showed that the group $\langle a, b \mid a^3 = b^3 = (ab)^3 = \emptyset$, exponent $9\rangle$ has a largest finite quotient of order 243 . It is easily shown by coset enumeration that $\langle a, b \mid a^3 = b^3 = (ab)^3 = (aab)^9 = \emptyset\rangle$ is a presentation for this finite group, and in fact a minimal presentation.

By applying the Canberra nilpotent quotient program we have shown that the restricted Burnside problem is answered affirmatively for the freest group of exponent 9 generated by two elements of order 3 whose commutator has order 3 . Thus the nilpotent quotient program shows that a preimage of

$$\langle a, b \mid a^3 = b^3 = [a, b]^3 = \emptyset, \text{ exponent } 9\rangle$$

has a largest finite quotient of class 18 and order 3^{71} .

The unrestricted problem has been solved for a quotient of the free group considered above, namely $\langle a, b \mid a^3 = b^3 = [a, b]^3 = (ababb)^3 = \emptyset$, exponent $9\rangle$. The nilpotent quotient program shows that this group has a largest finite quotient of class 10 and order 3^{12} . The nilpotent quotient program provides three ninth powers $\{(ab)^9, (aab)^9, (aabab)^9\}$ which suffice to define this group as a group of 3-power order when added to the four explicitly given third powers. Enumeration of the cosets of $\langle ab\rangle$ in

$$\langle a, b \mid a^3 = b^3 = [a, b]^3 = (ababb)^3 = (ab)^9 = (aab)^9 = (aabab)^9 = \emptyset\rangle$$

yields index 59049 , completing the solution of the unrestricted problem in this case.

We have applied the nilpotent quotient program to the freest group of exponent 9 generated by two elements of order 3 whose product has order 3 . Computations to date reveal that it has a preimage whose class 18 quotient has order 3^{603}

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