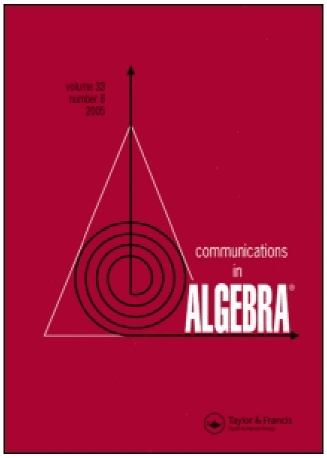

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Commutators in groups expressed as products of powers

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COMMUTATORS IN GROUPS EXPRESSED AS PRODUCTS OF POWERS

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Abstract

It is well known that in a free group the simple commutator $[Y, X]$ (alternatively called the first Engel word) can be expressed as a product of squares. Likewise the second Engel word $[Y, X, X]$ can be expressed as a product of cubes. Results on groups of exponent four imply that the fifth Engel word $[Y, X, X, X, X, X]$ can be expressed as a product of fourth powers, and explicit expressions have now been obtained with the assistance of a computer. The results and the computer-aided technique are described.

1. Introduction

Let G be a group and let X, Y be elements of G. We define the simple commutator of Y and X, $[Y, X]$, by $[Y, X] = Y^{-1}X^{-1}YX$, and we define the nth Engel word of Y and X, $[Y, X; n]$, inductively by $[Y, X; 0] = Y$ and $[Y, X; n+1] = \big[[Y, X; n], X\big]$.

In a free group (and hence in any group) the first Engel word can be expressed as a product of squares and the second Engel word can be expressed as a product of cubes. In the 2-generator Burnside group of exponent 4, $B(2, 4)$, the fifth Engel identity holds, that is, $[Y, X; 5] = e$. This may be verified by direct computation using, for example, a consistent power commutator presentation for $B(2, 4)$ provided by a nilpotent quotient algorithm (see Newman [8]). It follows that in a free group the fifth Engel word can be expressed as a product of fourth powers.

In the 2-generator restricted Burnside group of exponent 5, $\overline{B}(2, 5)$, the sixth Engel identity holds. This is proved by Kostrikin (see [4]) and may be verified by direct computation using the consistent power commutator presentation for $\overline{B}(2, 5)$ published in Havas, Wall and Wamsley [3]. It follows that if the (unrestricted) Burnside group $B(2, 5)$ is finite, then the sixth Engel word can be expressed as a product of fifth powers in a free group.

As early as 1960 Kostrikin [4] pointed out that expression of $[Y, X; 6]$ as a product of fifth powers has particular relevance to the Burnside problem for exponent 5. In 1973 [5] he suggested that expression of the fifth Engel word as a product of fourth powers could provide some ideas for expressing the sixth Engel word as a product of fifth powers.

In this paper we present an expression for the fifth Engel word as a product of fourth powers. The expression is obtained by use of a computer-aided method for proving relations in groups, described by Leech in [7], and based on coset enumeration.

One outcome of this work is the availability of computer programs for proving relations in groups. In particular these programs may be used to prove any relation having a formulation which specifies that a word in a finitely presented group lies in a subgroup of finite index.

2. The first and second Engel words

Direct expansion reveals that, for example,

$$[Y, X] = Y^{-2}\left(YX^{-1}\right)^{2}X^{2} , \tag{1}$$

$$[Y, X, X] = (YX)^{-3[Y,X]}(XYY)^{3YX}Y^{-3X} . \tag{2}$$

The derivation of these identities merits explanation. Identity (1) can easily be obtained by simply "fiddling". Identity (2) is readily obtained from the proof in Hall [1, p. 321] that $[Y, X, Y] = e$ in groups of exponent 3 .

In general it is not possible to express the first Engel word as a product of fewer than three squares. Likewise the second Engel word cannot be expressed as a product of fewer than three cubes. This may be shown, for example, by considering the

3-covering group of $B(2, 3)$. In this covering group, no single cube and no product of two cubes is equal to a second Engel word. This also may be verified using a consistent power commutator provided by a nilpotent quotient algorithm.

3. Computer methods

Leech's method for relation proof in groups [7] utilizes the working involved in the enumeration of the cosets of a subgroup. With only slight modification his method can be used to express $[Y, X; 5]$ as a product of fourth powers. The result comes from the working of coset enumerations for subgroups of $B(2, 4)$, using a presentation for $B(2, 4)$ which involves only fourth power relations.

Given a coset enumeration which shows that a subgroup is of finite index in a certain group, Leech's method provides a technique for expressing a word in the group generators which is in the subgroup as a word in terms of the generators of the subgroup. Leech does this by taking a circuit in the Schreier diagram and eliminating all deduced edges. Leech replaces deduced edges by deduction words. For our purposes here we need a little more. We need to keep a record of relators used, so we replace a deduced edge by a formal product of the relator used and the deduction word. Cancellation over a relator is allowed, at the expense of conjugation of that relator.

Programs which provide a computer implementation of these methods are available from the author. They are written in

FORTRAN, with only minor extensions beyond the 1966 ANSI standard, and are easily portable.

This modified process leads to the expression of a word in the subgroup as a product of subgroup generators and conjugates of relators. Using this method we can express $[Y, X; 5]$ as a product of fourth powers by following the working of the enumeration of the cosets of the trivial subgroup in $B(2, 4)$.

Leech's method was used by Havas [2] in obtaining a proof that the Fibonacci group $F(2, 7)$ is cyclic of order 29 . In that work it is shown that selection of the specific coset enumerations on which to base the proof could have a significant bearing on the length of proof. Similar principles apply here and other considerations also arise.

First, we must resolve the question of which presentation for $B(2, 4)$ we should use. Nine relator presentations exist for $B(2, 4)$ (see Leech [6]), however we have elected to use the ten relator presentation

$$B(2, 4) = \langle A, B; A^4, B^4, (AB)^4, (A^{-1}B)^4, (AB^2)^4, (A^2B)^4, [A, B]^4, (A^2B^2)^4, (A^{-1}BAB)^4, (AB^{-1}AB)^4\rangle$$

because it leads to coset enumerations involving fewer redundant cosets than the corresponding enumerations for the "more natural" nine relator presentations. The method used implies that the eventual expression for the Engel word includes only conjugates of relators explicitly given in the presentation, or their inverses. (See also the comment in section 5 on presentations for $B(2, 3)$.)

Second, we must choose the subgroup whose cosets are to be enumerated. It turns out that this choice interacts strongly with our third consideration, the choice of word in the subgroup which is to be expressed as a product of subgroup generators and conjugates of relators. Initially we comment on each of these separately, then tabulate the interaction between them.

Using our method, the enumeration of cosets of a subgroup of $B(2, 4)$ provides for a word in the subgroup an expression as a product of subgroup generators and conjugates of relators. However this expression need not be in its simplest form, even as far as subgroup generators are concerned. Thus, for an enumeration over a non-trivial subgroup when the word to be expressed is trivial, there may well be non-trivial subgroup generators involved, but of course their product is trivial. It is possible to convert such an expression into one involving only conjugates of fourth powers by collecting together the subgroup generators, conjugating already present fourth powers. Then the product of the subgroup generators can itself be expressed as a product of fourth powers. As long as the subgroup is not too complicated this can often be done by eye. Thus words can be expressed in terms of fourth powers using enumerations over various subgroups of $B(2, 4)$.

Our aim is to obtain an expression for $[Y, X; 5]$ as a product of fourth powers. This suggests that, in the context of our chosen presentation for $B(2, 4)$, we should express $[B, A; 5]$ in terms of the generators of various subgroups of $B(2, 4)$. However there are automorphic images which would also suffice, and even other related words, such as $\big[[B, A], B; 4\big]$ and its automorphic images, could merit consideration.

In the following table we present statistics on the performance of our methods in expressing various Engel words in terms of the generators of various subgroups.

SUBGROUP	$\langle B, (AB)^2\rangle$	$\langle AB\rangle$	$\langle A\rangle$	$\langle B\rangle$	$\langle e\rangle$	$\langle BA\rangle$	$\langle [A, B]\rangle$
Index	64	1024	1024	1024	4096	1024	1024
Total	64	1033	1046	1048	4124	1032	1062
$[B, A; 5]$	644	794	638	997	701	308	588
$[A, B; 5]$	715	557	1172	624	1446	1372	730
$[B^{-1}, A; 5]$	701	765	508	864	763	640	1001
$[A^{-1}, B; 5]$	877	1113	1298	462	1049	1143	767
$[B, A^{-1}; 5]$	733	707	822	760	624	665	832
$[A, B^{-1}; 5]$	803	932	1348	702	1297	905	1092
$[B^{-1}, A^{-1}; 5]$	622	436	620	840	802	493	981
$[A^{-1}, B^{-1}; 5]$	1125	1009	1304	584	996	632	867

SUBGROUP	$\langle BA^{-1}\rangle$	$\langle B^{-1}A\rangle$	$\langle B^2A^2\rangle$	$\langle B^A\rangle$	$\langle A^2, B^2\rangle$	$\langle (BA)^2\rangle$	$\langle (AB)^2\rangle$
Index	1024	1024	1024	1024	512	2048	2048
Total	1060	1054	1080	1044	552	2080	2080
$[B, A; 5]$	752	819	1310	496	1110	305	669
$[A, B; 5]$	1166	890	770	1415	635	919	431
$[B^{-1}, A; 5]$	912	600	1013	1015	444	548	940
$[A^{-1}, B; 5]$	1409	521	930	1863	1016	1502	807
$[B, A^{-1}; 5]$	581	730	688	978	1165	714	473
$[A, B^{-1}; 5]$	457	889	1223	1441	551	1158	1165
$[B^{-1}, A^{-1}; 5]$	805	872	623	1047	443	753	358
$[A^{-1}, B^{-1}; 5]$	805	916	874	1209	803	481	1035

The top row of each part of the table indicates the subgroup over which the coset enumeration was performed, the second row gives the index of that subgroup in $B(2, 4)$, and the third row indicates the total number of cosets defined during the enumeration. The remaining rows indicate the number of deduced edges actually replaced by our program in the process of converting the circuit around the specified Engel word into a form without deduced edges.

(It should be noted that this number is not the exact number of fourth powers which would be obtained in an expression for the corresponding Engel word based on this enumeration. The count may include replacements of deductions which arise from coincidence words which do not involve explicit use of a relator and replacements of deductions which arise in the definition of the subgroup. Also, even though the Engel word is trivial in $B(2, 4)$, the expression obtained may include a product of subgroup generators which requires a few more fourth powers to show that this product is trivial. However the number given does provide a good guide to the number of fourth powers in a consequent expression for the Engel word.)

There is a perhaps surprising amount of variation in the number of deduced edges replaced. The table speaks for itself.

Since there are two instances where the number of replacements is distinctly lower than any of the rest, we focus attention on those. The best figure is 305 replacements for $[B, A; 5]$ in the index 2048 coset enumeration over $\langle (BA)^2 \rangle$, with 308 replacements for $[B, A; 5]$ in the index 1024 coset enumeration over $\langle BA \rangle$ coming a close second.

For a given Engel word and a given coset enumeration we can still cause variation in the number of replacements. Our program replaces deduced edges in the reverse order to which the deductions were made. This is particularly convenient because it means that the deduction replacement rules need be scanned through only once. However it is possible to vary the order of replacement, and such variation does have an effect on the number of replacements made.

Consider the 305 replacements made during the expression of $[B, A; 5]$ in terms of $(BA)^2$ using the coset enumeration over $\langle (BA)^2 \rangle$. There are 2116 deduction rules. Interchange of the order of the first two deductions made (that is, an exchange of the order of the last two rules used by our program) causes the number of replacements to be reduced to 293 . This retains the desirable feature that the rules need only be scanned through once, without backtracking. The number of possible orders for using the rules is astronomical, and we have made no serious attempt to implement a method for selecting an optimal order.

The explicit expression ultimately obtained for the Engel word $[B, A; 5]$ in terms of fourth powers using the index 2048 enumeration over $\langle (BA)^2 \rangle$ is a product of 290 fourth powers, while the expression obtained using the index 1024 enumeration over $\langle BA \rangle$ is a product of 296 fourth powers. However this is not the end of the story. Included in these expressions are some fourth powers together with their inverses. In the case of the 290 fourth power expression there are 19 (fourth power, inverse) pairs which may be removed by conjugation. In the case of the 296 fourth power expression there are 23 such pairs. Thus the 296 fourth power expression may be readily converted to a 250 fourth power expression by conjugating out these pairs. This is the smallest number of fourth powers that we have found which provide an expression for a fifth Engel word.

4. The fifth Engel word

In this section we present an expression for the fifth Engel word $[B, A; 5]$ as an explicit product of 296 fourth powers, derived from the working of a coset enumeration for the subgroup $\langle BA \rangle$ in $B(2, 4)$. This expression implicitly contains an expression for this Engel word in terms of 250 fourth powers.

We have adopted a number of notational conveniences, devices for typographical ease. "A^{-1}" is replaced by "C" and "B^{-1}" by "D". The exponents on each of the words (4, of course) is omitted. Each power is separated from the next by a space. A dot "." is used to represent conjugation; thus "$A.BA$" stands for "A^{BA}". Matched (fourth power, inverse)-pairs are delimited by brackets, "(" preceding the first occurrence and ")" following the second occurrence. Thus such pairs may be removed from the expression by conjugating all the powers in between by the appropriate fourth power.

$[B, A; 5]$ =
(C D.ADCCB BC.CCB A.B D.CBAB CDAB.B D (DCC.CDADD (BC.D (A.BCBCD
(A.BCD (A.DABCD (D.CD (BC.DDCD (D.CDDCD (BAB (DC (C.BAABA
(AAB.CCBAABA (DA.AABA (CBAD.DADAAABA (B.ADAAABA BAB.ABCBADAAABA
DC.CBADAAABA A.BADAAABA CBAD.ABADAAABA C.DAAABA CDD.DCBCDAAABA
B.CBCDAAABA AD.DAAABA B.ADDAAABA CD.DDAAABA CD.DCDDDAAABA
D.ADAAABA) BCDA.DADAAABA) CB.AABA) DCC.CCBAABA) A.BAABA) AB) D.AB
BAB.ABCDADDAB C.DDAB AAB.CDDAB BA.DAB DCD.AADAB D BC.DD D.CDD BBA
DC.BA CD C.DCD C.BCCCDCD AAB.CCBCCCDCD AD.CCDCD CBAD.DADCCDCD
B.ADCCDCD BADC.CDDCDDDCCDCD A.DDCDDDCCDCD DCC.CDADDCDDDCCDCD
A.DADDCDDDCCDCD BC.DCDDDCCDCD A.BCDCDDDCCDCD D.ADCCDDDCCDCD
BC.CCDDDCCDCD A.DDDCCDCD D.CBADDDCCDCD BCDA.DDCCDCD DCC.CDADDCCDCD
BC.DCCDCD A.BCBCDCCDCD A.BCDCCDCD AD.DABCDCCDCD B.ABCDCCDCD

DCD.CDCCDCD AB.CDCD B.CDABCDCD DC.CDABCDCD A.DABCDCD D.CDCD
BC.DDCDCD D.CDDCDCD BAB.CD DC.CD C C.BCC AAB.CCBCC AD.C CBAD.DADC
B.ADC DA.ADC C.DC B.ADADCCDC CD.DADCCDC CD.DDADCCDC B.ADCCDC
ABCD.CDC AB DCD) B.CDDCD) AD.DDCD) B.CD) C.DABCD) C.BCD) C.BCBCD)
AD.D) AAB.CDADD) C.DD AAB.CDD CB.BCDD D.CDD D.ABBA BBA.DDABBA
D.CDABBA BA.DDCDABBA CD.DCDABBA BAB.CDCDDCDABBA A.DDCDABBA
CBAD.AADDCDABBA D.ADADDCDABBA BC.DCDABBA BA.ABBA DCD.AABBA
DCD.CDAABBA BA.AABBA CCD.ABBA A.DABBA D.A BC.DA D.CDA BAB.A
DCBA.BABBABA AB.BADCBABABBABA B.CBABABBABA B.CBCBABABBABA DA.ABABBABA
B.AABABBABA DA.AABABBABA C.BABBABA CDAB.BBABA D.ABA DCC.CDADABA
CB.BA A.BCBBA DCD.CDCDCCDCCCBBA AB.CDCCCBBA AB.CCBBA B.CDABCCBBA
DCC.CCBBA A.BBA DC.CCDCCBA A.DCCBA A.BA CB.ABA BAB.CDCCBCBABA
BADC.CCBCBABA DA.ADCCCBCBABA C.DCCCBCBABA C.BAABCBCBABA
AAB.CCBAABCBCBABA B.ADCBAABCBCBABA CDAB.ABCBCBABA C.BABCBCBABA
C.DCCBABCBCBABA DADC.CBABA C.DCDADCCBABA AB.CCDCDADCCBABA
AD.CDCDADCCBABA C.DCCBABA AB.CCDCCBABA C.BCCDCCBABA C.DCCBCCDCCBABA
D.CCDCCBABA C.DDCCDCCBABA C.BCCDDCCDCCBABA AAB.CCBCCDDCCDCCBABA
AD.CDDCCDCCBABA BABC.DCCDCCBABA (AAB.BABA (CD.ABBABA DCD.CCDABBABA
A.DABBABA (D.ABA (BC.DABA (D.CDABA (BAB.ABA (DC.BABABA C.BABABABA
C.BCCBABABABA AAB.CCBCCBABABABA AD.CBABABABA CBAD.DADCBABABABA
B.ADCBABABABA B.CBABABABA C.BCDCBCBABABABA CD.DABBABABABA
DCC.CDDABBABABABA A.DDABBABABABA DCD.ABCDADDABBABABABA B.ABBABABABA
DC.BABABABA C.BABABABABA C.BCCBABABABABA AAB.CCBCCBABABABABA
AD.CBABABABABA CBAD.DADCBABABABABA B.ADCBABABABABA
BA.DCDDDCBABABABABA BA.DDCBABABABABA D.ADDCBABABABABA
BC.DCBABABABABA D.CBCDCBABABABABA BBA.DCBCDCBABABABABA
C.DCBCDCBABABABABA ADAB.CCDCBCDCBABABABABA DABA.ABABABABA
C.BADABAABABABABA ABAD.CCBADABAABABABABA D.ADABAABABABABA
D.AABABABABA BC.DDAABABABABA D.CDDAABABABABA CD.DCDDAABABABABA
BBA.AABABABABA DC.BAAABABABABA CD.DCBAAABABABABA D.AAABABABABA
D.ADAAABABABABA BCDA.DADAAABABABABA CB.AABABABABA DCC.CCBAABABABABA
A.BAABABABABA AB.BABABA) DCD.ABA) B.CDABA) AD.DABA) B.ABA)
BAB.AADABBABA ABA.DADABBABA CD.DADABBABA BAB.CDDADABBABA
DDCC.CCDDADABBABA A.DDADABBABA B.ADABBABA CBAD.DDCDABBABA
B.CDABBABA DCD.CDABBABA BA.ABBABA) DCC.BABA) AB.CDCCBABA

DCC.CDCCBABA A.DCCBABA BC.CCBABA A.BCBCCCBABA AB.CDABCBCCCBABA A.BADCBBCCCBABA CD.DCBBCCCBABA D.CCCBABA D.CCBABA BBA.DDCCBABA B.CBADDCCBABA C.DDCCBABA AD.CCDDCCBABA C.DCCDDCCBABA C.DCCBABA C.BCCDCCBABA AAB.CCBCCDCCBABA D.CCBCCDCCBABA C.DDCCBCCDCCBABA D.CCDCCBABA C.DCCDCCBABA AAB.BABA D.AABBABA D.AADAABBABA BBAA.DDAADAABBABA AB.CDDAADAABBABA CB.ADAABBABA D.ADAABBABA CB.ABBABA D.CBABBABA CDAB.BBABA D.ABA DCC.CDADABA CB.BA A.BCBBA A.BBA BAB.ABCDCDCCDABBA D.ABABCDCDCCDABBA D.CDCDCCDABBA AB.CDABBA D.CDABBA DC.BA BAB.CDCBA D.ABCDCBA BC.DABCDCBA A)

The first five fourth powers in the above expression, written in a more conventional notation, are:

$$A^{-4} \quad B^{-4AB^{-1}A^{-2}B} \quad \left(BA^{-1}\right)^{4A^{-2}B} \quad A^{4B} \quad B^{-4A^{-1}BAB} \,.$$

5. In retrospect

We now have expressions for the fifth Engel word as a product of fourth powers. What use are they and how good are they?

It seems unlikely that these expressions will provide any assistance towards expressing the sixth Engel word as a product of fifth powers. The method, relying on coset enumeration as it does, effectively would need a finiteness proof for $B(2, 5)$ as a prerequisite. The result, some 250 fourth powers for the fifth Engel word, suggests that if the sixth Engel word can be expressed as a product of fifth powers then the expression is likely to be rather nasty.

It is reasonable to suggest that the given expression for the fifth Engel word as a product of fourth powers is not unduly complex because of the good performance of the same method in expressing shorter Engel words as products of powers. We made a study of computer expressions for the first Engel word as a product of squares and for the second Engel word as a product of cubes, using this method. This study reveals that, for the first Engel word, computer expressions based on a reasonable range of coset enumerations and various Engel words always required only three squares, the minimum possible. For the second Engel word expressions in terms of from 3 to 15 cubes were obtained by computer, with the minimum possible, 3 , again being attained.

The search for expressions for the second Engel word alerted us to one effect of the choice of presentation for the group relative to which cosets are enumerated. Initially we performed enumerations in $B(2, 3)$ using the "natural" shortest presentation:

$$B(2, 3) = \langle A, B; A^3, B^3, (AB)^3, (A^{-1}B)^3 \rangle .$$

With this presentation we never attained an expression with fewer than 4 cubes, over a range of attempts. The reason for this is now clear: it can be shown that no product of conjugates of 3 of these relators and their inverses is equal to the second Engel word. After noting this we switched to the presentation:

$$B(2, 3) = \langle A, B; A^3, B^3, (AB)^3, (A^2B)^3 \rangle .$$

With this presentation we obtained expressions involving exactly 3 cubes in a number of ways.

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