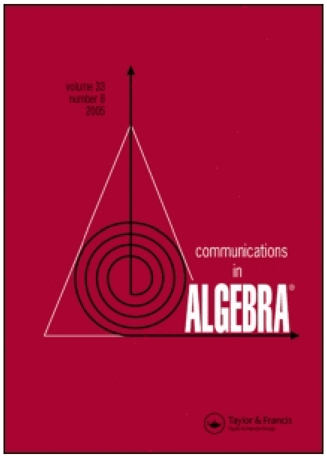

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Minimal presentations for finite groups of prime-power order

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MINIMAL PRESENTATIONS FOR FINITE GROUPS OF PRIME-POWER ORDER

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In his survey "Minimal presentations for finite groups" Wamsley [8, Question 14], asks: "Are there any four generator-five relation finite groups?" A more precise formulation is: are there any finite groups which can be generated by 4 elements but not by 3 elements and which can be defined by 5 relations on a 4-element generating set? The answer is yes. We describe four such groups and indicate how they were found. The groups are given in order of increasing size. The proof for the first is simply a (huge) coset enumeration. The other proofs are more interesting; only one is spelt out.

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Consider the group G which is generated by $\{a, b, c, d\}$ and is defined by the relations

$$
\begin{aligned}
a^2 &= [d, c][b, a] \\
b^2 &= [c, b] \\
c^2 &= [d, c] \\
d^2 &= [d, a][c, b][b, a] \\
[d, b] &= [c, a] .
\end{aligned}
$$

Clearly the commutator quotient G/G' is elementary abelian of order 16 so G cannot be generated by 3 elements. Thus it remains to show that G is finite. In theory the simplest, though not necessarily the shortest, way to prove the finiteness of a group given by a finite presentation is coset enumeration over the identity subgroup. In this case, using a reasonably sophisticated computer program (a descendant of that described by Cannon, Dimino, Havas and Watson [1]) with enough computer memory to define some 250 000 active cosets, the approach works. Coset enumeration over the trivial subgroup yields that G has order 2^{16} .

Now consider the group H which is generated by $\{a, b, c, d\}$ and is defined by the relations

$$
\begin{aligned}
a^2 &= [d, c][b, a] \\
b^2 &= [c, b] \\
c^2 &= [d, c]
\end{aligned}
$$

$$d^2 = [d, a][c, b][b, a][d, c]$$

$$[d, b] = [c, a] .$$

The commutator quotient H/H' is elementary abelian of order 16 so H cannot be generated by 3 elements and again it remains to show that H is finite. In this case coset enumeration over the trivial subgroup cannot be completed with the resources available to us. However coset enumeration over the subgroup A generated by a does complete and yields that A has index 2^{13} in H. With another computer-aided calculation this suffices to prove that H is finite, as follows.

Let K be the largest normal subgroup of H contained in A. Then K is cyclic and has finite index in H. Let S be the centraliser of K in H. Then S contains H' and the centre of S has finite index in S. Hence (Robinson [7], p. 102), the commutator subgroup of S is finite. The elements of finite order in S therefore form a finite subgroup T and S/T is torsion-free abelian. Thus H/T is an extension of a torsion free abelian group by an elementary abelian group of order dividing 16. Information provided by another computer program, the nilpotent quotient algorithm (see Newman [6]), shows that H has no quotient group of 2-power order exceeding 2^{17}. Hence H/T and, consequently, H is finite. A little more computing gives the order of H. Let N be the normal subgroup of H which has H/N of order 2^{17}. It is easy to see from the presentation for H/N obtained from the nilpotent quotient algorithm that aN has order 16 so AN has index 2^{13} in H. Hence A contains N. Therefore N is cyclic of odd order. Further coset enumerations over the subgroups B, C, D (generated by b, c, d respectively) yield, by a similar argument, that N is contained also in B, C, D. Hence N is central in H. So H is nilpotent and, since H/H' has 2-power order, H has 2-power order which must be 2^{17}.

Similar arguments show that the group generated by $\{a, b, c, d\}$ and defined by the relations

$$\begin{aligned} a^2 &= [d, c][b, a] \\ b^2 &= [c, b] \\ c^2 &= [d, c] \\ d^2 &= [d, a][c, b][b, a][c, a] \\ [d, b] &= [c, a] \end{aligned}$$

is finite of order 2^{18}, and that the group generated by $\{a, b, c, d\}$ and defined by the relations

$$\begin{aligned} a^2 &= [d, c][b, a] \\ b^2 &= [c, b] \\ c^2 &= [d, c] \\ d^2 &= [d, a][c, b][b, a][c, a][d, c] \\ [d, b] &= [c, a] \end{aligned}$$

is finite of order 2^{19} .

Observe that these groups are groups of prime-power order, so, by the Golod-Šafarevič Theorem (Johnson [3], p. 156), need at least 5 relations to define them.

Calculations of the sort used above yield little more than the information stated. Other calculations, such as those described by Leech [4], might be used in an attempt to gain more understanding. It seems likely that a good deal more effort would be required to obtain any useful insight.

The presentations above have been arrived at by a process of evolution. A natural starting point was the work of Mennicke [5] which exhibited for the first time finite groups which have a (non-trivial) 3-generator 3-relation presentation. These suggest for 4 generators that one might consider the groups $K(u)$ generated by $\{a, b, c, d\}$ and defined by the 6 relations

$$a^b = a^u$$

$$b^c = b^u$$

$$c^d = c^u$$

$$d^a = d^u$$

$$[a, c] = [b, d] = e \quad \text{(the identity)}.$$

For $|u| \geq 2$ these groups are finite (a simplified version of Mennicke's proof suffices). We are indebted to B.H. Neumann for first showing us these presentations; they also appear in Johnson ([3], pp. 49-50). For $u = 1 \pm p^k$ where p is a prime and

$p^k \neq 2$, the group has a largest p-quotient which is nilpotent of class 2 and has order p^{8k} . Coset enumeration shows that K(3) has order 2^{12} and K(-2) has order 3^8 .

Replacing the last two relations for K(u) by the single relation [a, c] = [b, d] gives a 4-generator 5-relation presentation. The nilpotent quotient algorithm yields the following table.

u	order of the largest p-quotient
3	2^{22}
-2, 4	3^{16}
-4, 6	5^{15}
-6, 8	7^{15}
-10, 12	11^{15} .

These largest p-quotients are interesting because they are either 4-generator 5-relation finite groups or they are finite nilpotent groups whose deficiency exceeds their multiplicator rank (Wamsley [8], Questions 9 and 11). However we have not been able to decide which is the case here. Similar groups for all odd p are given in Wisliceny [9] and [10]; the nilpotent quotient algorithm shows that these have order p^{15} for small odd primes (this presumably holds for all odd primes).

It is not difficult to see that a 4-generator 5-relation group of prime-power order has order at least p^{14} . A simple alteration to the above 5-relation presentation, replacing the last relation by $[a, c] = [b, d][a, b]$, gives a presentation whose group has largest p-quotient of order p^{14} when $u = 1 \pm p$ with p odd. The nilpotent quotient algorithm shows that when $u = 3$ the group has a largest 2-quotient of order 2^{20} . If the nilpotent groups so obtained are defined by the given presentation, it is reasonable to hope that methods based on coset enumeration might be successful in proving this for the "small" orders (as they were for similar size problems reported in Grunewald, Havas, Mennicke and Newman [2]). This has not been the case. However, as our main examples show, suitable modifications do work.

We have investigated modified presentations obtained in a similar way, that is by incorporating into the relations various additional commutators. Of course the number of possible modifications is vast - we selected a few promising ones for testing. The chosen presentations were first checked with the nilpotent quotient algorithm for $p = 11$. If the largest 11-quotient had order 11^{14} or 11^{15} , then the corresponding presentation for $p = 2$ was tested. Where the largest 2-quotient had order at most 2^{19} , the resulting presentation was modified via Tietze transformations to a form which experience suggested was more likely to lead to successful coset enumeration. As well as providing the four 5-relation presentations for finite groups given, this procedure has yielded presentations for which we have no finiteness proof. In particular we have found presentations whose largest 2-quotients have order 2^{15} where coset enumerations have suggested that the groups so defined might well be infinite.

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