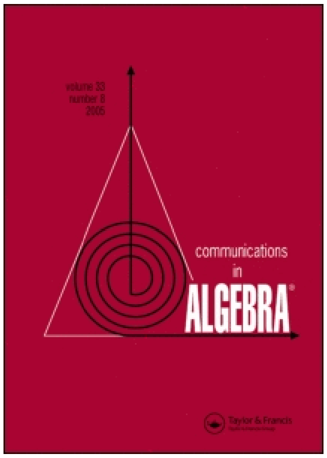


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## Groups of exponent five and class four

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# GROUPS OF EXPONENT FIVE AND CLASS FOUR

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## Abstract

We investigate presentations for the freest two-generator group of exponent five and class four, and obtain a number of minimal presentations, including two which contain the least possible number of non-fifth-power relators. Our aims are threefold: firstly, to





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provide some partial evidence in favour of the finiteness of the Burnside group of exponent five on two generators; secondly, to examine a refinement of the well-known question concerning the existence of minimal presentations for finite $p$-groups; and thirdly, to illustrate several ways in which a number of available computer techniques can be combined to demonstrate the finiteness of a group with a given presentation.

## 1. Minimal and optimal presentations

Let $G$ be a finite $p$-group minimally generated by $d$ elements, and $G \cong F/R$ a presentation of $G$ as a quotient of a free group $F$ of rank $d$ by a normal subgroup $R$. A lower bound for the number of relators required in such a presentation, that is, for the size of a set of words with normal closure $R$ in $F$, is given by the rank $m$ of the $p$-multiplier

$$R/[R,\ F]R^p \cong H_2(G,\ \mathbb{Z}_p)$$

(see Gruenberg [4], §7.2). Following Wamsley [16], we shall call a $d$-generator, $m$-relator presentation for $G$ *minimal.* It is a well-known open question whether all finite $p$-groups have such a minimal presentation (Swan [15], Wamsley [16]).

Now suppose that $G$ is finite of exponent $q = p^e$ . Relators for $G$ may be categorized into $q$th powers and others, and presentations may be classified according to the numbers of each type of relator present. For example, we may ask: what is the smallest number of non-$q$th-power relators needed, together with $q$th powers, to present $G$ ?

If we are given a fixed minimal generating set for $G$ (that is, a fixed homomorphism from $F$ , of rank $d$ , to $G$ , with kernel $R$ say), then a lower bound on the number of non-$q$th-power relators in the given generators required (together with $q$th powers) to define $G$ is given by the rank $s$ of

$$R/[R, F]R^pF^q ,$$

since this is the quotient of the $p$-multiplier obtained by making *all* $q$th powers trivial. We shall call a minimal presentation for $G$ which involves exactly $s$ non-$q$th-power relators, together with $m - s$ $q$th powers, an *optimal* presentation. More generally, a presentation consisting of $i$ non-$q$th-power and $j$ $q$th-power relators will be called an $(i, j)$ presentation for short.

The question of whether all finite $p$-groups possess an optimal presentation (unlike the corresponding question for minimal presentations) admits a ready negative answer. For let $B = B(d, p)$ be a $d$-generator Burnside group of exponent $p$ , and $\overline{B}$ ($\cong F/R$) its largest finite quotient: the existence of $\overline{B}$ is guaranteed by Kostrikin's positive solution to the restricted Burnside problem [11]. Since $F/[R, F]F^p$ is a finite group of exponent $p$ , we have $[R, F]F^p = R$ , and in this case $s = 0$ . So an optimal presentation for $\overline{B}$ would consist entirely of $p$th-power relators. A presentation for $\overline{B}$ with this last property exists if and only if $B = \overline{B}$, or equivalently, if and only if $B$ is finite. In particular, if $d \geq 2$ and $p > 665$, so that by the celebrated result of Novikov and Adian (see [1]) $B$ is infinite, then $\overline{B}$ has no optimal presentation.

However, the question of the existence of optimal presentations for finite groups of *small* prime-power exponent remains open, and here a number of positive results are available. For example, coset enumeration can be used to show that the three groups $B(3, 3)$, $B(2, 4)$, and

$$I(3, 4) = \langle a, b, c \mid a^2, b^2, c^2; \text{ exponent } 4\rangle$$

have $(0, 13)$, $(0, 9)$, and $(3, 7)$ presentations respectively (Leech [12]). The nilpotent quotient algorithm or NQA (Macdonald [13], Bayes, Kautsky and Wamsley [2], Havas and Newman [8]) permits computer calculation of and in the $p$-multipliers of these groups, and shows that the presentations mentioned are not only minimal as pointed out by Macdonald [13] but optimal as well. Havas and Newman [8] give minimal presentations for a group of order $2^{19}$ with exponent four, and for exponent nine groups of order $3^5$ and $3^{12}$, while Grunewald, Havas, Mennicke and Newman [5] give a minimal presentation for a group of exponent eight and order $2^{21}$; all of these presentations are in fact optimal.

Further study of the question of which finite $p$-groups have an optimal presentation is relevant in two ways to the Burnside problem, and particularly to its first undecided case, that of exponent five.

Firstly, the disparity in the above results between the case of the small exponents $3$ and $4$ and the case of large prime exponents parallels, at least superficially, the dichotomy in the answer to the Burnside problem (positive for the former exponents, negative for the latter). So an investigation into the existence of optimal presentations for groups of exponent five may indirectly shed some light on the Burnside problem in that exponent.

Secondly, the computational techniques applicable in a search for optimal presentations are among those which would presumably be involved in a computer-aided attempt to prove $B(2, 5)$ finite (for example along the lines suggested by M. Hall, Jr. [6]). The order of $\overline{B}(2, 5)$, namely $5^{34}$ (see [9]), gives a hint of the difficulties which can be expected in such an attempt. However, a manageable analogue of the full Burnside problem is obtained if we seek *optimal* presentations for quotients of $B(2, 5)$ of small class. In such a search, the evasion of the problem of size is to some extent balanced by the imposition of the requirement of optimality. Thus the techniques available may be put to an interesting and still feasible trial, in the course of which they may be expected to undergo further development and refinement.

Before discussing how computer techniques were successfully employed to obtain optimal presentations for the largest class-four quotient of $B(2, 5)$, we see how much can be said about optimal presentations of low-class quotients of two-generator Burnside groups for primes in general.

## 2. Results for an arbitrary prime

Let $F = \langle a, b \rangle$ be a free group on two generators, and define the following basic commutators in $F$ :

$$c_3 = [b, a] ,$$
$$c_4 = [c_3, a] , \quad c_5 = [c_3, b] ,$$
$$c_6 = [c_4, a] , \quad c_7 = [c_4, b] , \quad c_8 = [c_5, b] .$$

We begin by obtaining minimal presentations for the two-generator free nilpotent groups of class up to four.

**LEMMA:**

$$F/\gamma_2 F \cong \langle a, b \mid c_3 \rangle ;$$
$$F/\gamma_3 F \cong \langle a, b \mid c_4, c_5 \rangle ;$$
$$F/\gamma_4 F \cong \langle a, b \mid c_6, c_7, c_8 \rangle ;$$
$$F/\gamma_5 F \cong \langle a, b \mid [c_i, a], [c_i, b] \ (i = 6, 7, 8) \rangle .$$

Proof: The only difficult case is class three, where it must be shown that $c_6 = c_7 = c_8 = 1$ implies $[c_5, a] = 1$ ; we are indebted to J.R.J. Groves for the following short proof. We work in the group

$$G = \langle a, b \mid c_6, c_7, c_8 \rangle ,$$

and utilize the well-known commutator identities

$$\begin{aligned}[x, yz] &= [x, z][x, y]^z \\ &= [x, z][x, y][x, y, z] .\end{aligned}$$

Now

$$[b, a, ba] = [b, a, a][b, a, b][b, a, b, a] ,$$

but also, as $[b, a, a] = c_4$ is central in $G$ ,

$$\begin{aligned}[b, a, ba] &= \big[b, a, ab[b, a]\big] \\ &= [b, a, ab]^{[b,a]} \\ &= ([b, a, b][b, a, a][b, a, a, b])^{[b,a]} \\ &= [b, a, a][b, a, b]^{[b,a]} .\end{aligned}$$

So we have

$$\begin{aligned}[b, a, b, a] = [b, a, b]^{-1}[b, a, b]^{[b,a]} &= \big[b, a, b, [b, a]\big] \\ &= \big[b, a, b, (b^{-1}a^{-1}b)a\big] \\ &= [b, a, b, a]\big[b, a, b, b^{-1}a^{-1}b\big]^a .\end{aligned}$$

We deduce that $\big[b, a, b, b^{-1}a^{-1}b\big] = 1$ , so that in $G$ , $b^{-1}a^{-1}b$ like $b$ centralizes $[b, a, b] = c_5$ . Hence $a$ also centralizes $c_5$ , as claimed.

It follows that $\langle c_4, c_5\rangle$ is central in $G$ . But

$$G/\langle c_4, c_5\rangle \cong \langle a, b \mid c_4, c_5\rangle \cong F/\gamma_3F$$

by the previous (easily proved) case. Hence $G$ is indeed nilpotent of class 3, so $G \cong F/\gamma_4 F$. The result for class 4 is now readily deduced.

For any prime $p$, let $B_p = F/F^p = B(2, p)$ denote the two-generator Burnside group of exponent $p$.

**THEOREM:** *The largest nilpotent quotients of $B_p$ of class up to four have the minimal presentations*

$$B_p/\gamma_2 B_p \cong \langle a, b \mid a^p, b^p, c_3 \rangle ,$$

$$B_p/\gamma_3 B_p \cong \langle a, b \mid a^p, b^p, c_4, c_5 \rangle \quad \text{if } p \geq 3 ,$$

$$B_p/\gamma_4 B_p \cong \langle a, b \mid a^p, b^p, c_6, c_7, c_8 \rangle \quad \text{if } p \geq 5 ,$$

$$B_p/\gamma_5 B_p \cong \langle a, b \mid a^p, b^p, [c_i, a], [c_i, b] \ (i = 6, 7, 8) \rangle \quad \text{if } p \geq 5 .$$

*Furthermore, each of these presentations is optimal, except when $p - 1$ is equal to the class of the quotient group.*

**Proof:** We shall prove the last assertion; the other cases are similar. If $G$ is the group given by the last presentation, then by the Lemma $\gamma_5 G = 1$. For $2 \leq i \leq 5$, $\gamma_i G/\gamma_{i+1} G$ has exponent $p$, since $G/G'$ does (Huppert [10], III 2.13 (b), p. 266). Moreover, $\gamma_i G/\gamma_{i+1} G$ is generated by the basic commutators of the appropriate weight; we conclude that $|G|$ divides $p^8$. But $G$ is a preimage of $B_p/\gamma_5 B_p$, and since $p \geq 5$

the order of this last group is exactly $p^8$ (Magnus, Karrass and Solitar [14], Section 5.7, Problem 9). It follows that $B_p/\gamma_5 B_p \cong G$ , as required.

For $p \geq 7$ , $B_p/\gamma_5 B_p$ has the exponent $p$ extension $B_p/\gamma_6 B_p$ , of order $p^{14}$ (Magnus, Karrass and Solitar, *loc. cit.*). It is thus clear that apart from the two power relators $a^p$ and $b^p$ , six non-power relators are required to span the $p$-multiplier of $B_p/\gamma_5 B_p$ , so the $(6, 2)$ presentation we have obtained is optimal. For $p = 5$ , the $p$-multiplier in question still has rank $8$ , so in this case the presentation found is at any rate minimal.

## 3. Results for exponent five and class four

In the remainder of the paper, we describe a computer-assisted search for optimal presentations of $B_5/\gamma_5 B_5$ , the freest two-generator group of exponent five and class four. Since $\gamma_5 B_5/\gamma_6 B_5$ has order $5^2$ , it is $(2, 6)$ presentations which are required.

Thus we are seeking two non-power and six power relators. First we consider likely candidates for the power relators. Here shortness seems a desirable criterion, both for aesthetic reasons and for ease of computation. To obtain at least six fifth powers which are distinct as relators, we must consider words of length up to three in the free group. There are (modulo cyclic permutation and inversion) eight distinct such words, namely

$$a,\ b,\ ab,\ ab^{-1},\ a^2b,\ a^2b^{-1},\ ab^2,\ ab^{-2}\ .$$

The first four of these, being shorter, seem suitable starters; two more are needed. Calculation with the NQA shows that the image of each of the sets $\{(a^2b)^5,\ (ab^{-2})^5\}$ and $\{(a^2b^{-1})^5,\ (ab^2)^5\}$ in the 5-multiplier of $B_5/\gamma_5B_5$ is linearly dependent, so these pairs are eliminated. This leaves four sets as candidates for the power relators of an optimal presentation, namely

$$S_1 = \{a^5,\ b^5,\ (ab)^5,\ (ab^{-1})^5,\ (a^2b)^5,\ (a^2b^{-1})^5\}\ ,$$

$$S_2 = \{a^5,\ b^5,\ (ab)^5,\ (ab^{-1})^5,\ (a^2b)^5,\ (ab^2)^5\}\ ,$$

$$S_3 = \{a^5,\ b^5,\ (ab)^5,\ (ab^{-1})^5,\ (a^2b^{-1})^5,\ (ab^{-2})^5\}\ ,$$

and

$$S_4 = \{a^5,\ b^5,\ (ab)^5,\ (ab^{-1})^5,\ (ab^2)^5,\ (ab^{-2})^5\}\ .$$

The next step is the selection of non-power relators. Here we took as a guide the minimal presentation

$$\left\langle a,\ b\ \mid\ a^5,\ b^5,\ [c_i,\ a],\ [c_i,\ b]\ (i = 6,\ 7,\ 8)\right\rangle$$

for $B_5/\gamma_5B_5$ obtained in the previous section, and attempted to replace non-power relators one at a time by power relators. This approach allowed us to build up useful experience by tackling a series of successively more difficult problems. The words

$[c_4, c_3]$ and $[c_5, c_3]$, which are trivial in $B_5/\gamma_5 B_5$ and shorter than those of the form $[c_i, a]$ and $[c_i, b]$ $(i = 6, 7, 8)$, also proved useful as non-power relators.

The power and non-power relators discussed above may be combined in various ways to yield a presentation $P$ which has the form of a minimal or optimal presentation for $B_5/\gamma_5 B_5$. Let $G$ be the group defined by $P$. It remains to be determined whether $G$ is indeed isomorphic to $B_5/\gamma_5 B_5$. We now describe our approach to this problem.

The presumed order of $G$, namely $5^8 = 390625$, precludes direct coset enumeration. Instead, we split the problem into two stages, by seeking a subgroup of $G$ for which both the order and the index may be computed. In detail, the procedure is as follows.

(a) A subgroup $H$, specified by generators expressed as words in $a$ and $b$, is selected. The index of $H$ in $G$ should be moderately small.

(b) The Reidemeister-Schreier method, implemented as described by Havas [7], is used to derive a presentation for $H$ from $P$.

(c) The presentation for $H$ is simplified by a Tietze transformation program. This program successively eliminates redundant generators (either chosen automatically by the program in accordance with some prespecified order of preference, or selected interactively by the user). After each elimination, the program searches for common subwords and shortens relators thereby, and uses any commutator relators present to collect together occurrences of like generators in other relators.

(d) If possible, the order of $H$ is determined from the simplified presentation, by a method appropriate to the expected structure of $H$ .

Our first choice for $H$ was the derived subgroup $G'$ . If indeed $G$ is isomorphic to $B_5/\gamma_5 B_5$ , then $G'$ is of index 25 in $G$ , and elementary abelian of order $5^6$ . Ideally, the Tietze program will reduce the original presentation for $G'$ , on 26 Schreier generators, to a presentation of $C_5^6$ on 6 generators, all visibly commuting and of order 5 . This reduction may be aided by the introduction of relators defining some or all of the elements $c_i$ in terms of the Schreier generators: if the program is then instructed to eliminate Schreier generators in favour of the $c_i$ , commutator relators involving the latter are soon deduced and result in rapid shortening of the presentation.

This method was successful in demonstrating that the first three presentations shown in the Table define $B_5/\gamma_5 B_5$ .

The best result that we were able to achieve via automatic elimination in reducing the number of non-power relators below four was a (3, 6) presentation. In order to dispense with the extra fifth power and to show that the group ($G_{35}$ say) defined by the (3, 5) presentation in the Table is $B_5/\gamma_5 B_5$ , the program requires assistance. It is necessary to hand pick the generators to be eliminated, with the aim of producing, or at least

TABLE

Minimal presentations of $B_5/\gamma_5 B_5$

| No. | Type | Non-power relators | Power relators | Method |
|---|---|---|---|---|
| 1 | (6, 2) | $[c_i, a]$, $[c_i, b]$ $(i = 6, 7, 8)$ | $a^5$, $b^5$ | $T$, $E$(4600) |
| 2 | (5, 3) | $[c_6, a]$, $[c_6, b]$, $[c_7, a]$, $[c_7, b]$, $[c_8, a]$ | $a^5$, $b^5$, $(ab)^5$ | $T$, $E$(24296) |
| 3 | (4, 4) | $[c_i, a]$, $[c_i, b]$ $(i = 6, 7)$ | $a^5$, $b^5$, $(ab)^5$, $(ab^{-1})^5$ | $T$, $E$(11664) |
| 4 | (4, 4) | $[c_6, a]$, $[c_8, b]$, $[c_4, c_3]$, $[c_5, c_3]$ | $a^5$, $b^5$, $(ab)^5$, $(ab^{-1})^5$ | $E$(12579) |
| 5 | (3, 5) | $[c_7, a]$, $[c_7, b]$, $[c_4, c_3]$ | $a^5$, $b^5$, $(ab)^5$, $(ab^{-1})^5$, $(a^2b)^5$ | $T$(with help) |
| 6 | (2, 6) | $[c_4, c_3]$, $[c_5, c_3]$ | $S_2$ | $E$(86769) |
| 7 | (2, 6) | $[c_4, c_3]$, $[c_5, c_3]$ | $S_3$ | $E$(4978) |

Methods: $T$ Tietze transformations in $G'$ .

$E$ Coset enumeration in $\langle b, a^{-1}ba, aba^{-1}\rangle$ (with total number of cosets defined shown in brackets).

preserving, as many commutator relators as possible. In this way one can obtain a presentation for $G'_{35}$ on six generators $x, y, \ldots$ , which includes relators indicating that $y$ commutes with all generators other than $x$ , and in addition the relators

$$x^{-1}yxy^4 , \quad x^{-1}yxy^{-1}xyx^{-1}y^{-1} .$$

The substitution of $y^{-4}$ for the common subword $x^{-1}yx$ transforms the second relator into

$$xyx^{-1}y^{-6} .$$

(Note that since this is *longer* than the second relator, such a substitution is not made by the Tietze program: these details illustrate a limitation in the program which, given the insolubility of the isomorphism problem, is essentially inherent.) It can now be deduced that $y^{25} = 1$ , and that $x$ normalizes $\langle y \rangle$ , so that $\langle y \rangle$ is normal in $G'_{35}$ . If the relator $y$ is added, the program quickly shows that $G'_{35}/\langle y \rangle \cong C_5^5$ . We now know that $|G_{35}|$ divides $5^9$ ; but the NQA demonstrates that the maximal 5-quotient of $G_{35}$ has order $5^8$ , so that $G_{35} \cong B_5/\gamma_5 B_5$ as desired.

Similar methods using only two non-power relators yielded nothing better than a (2, 14) presentation for $B_5/\gamma_5 B_5$ . Attempts to improve this result by applying coset enumeration (see [3]) to the presentations produced by the Tietze program were unsuccessful, probably because $G'$ is a six-generator group, so that storage per coset is large and the maximum number of cosets which can be defined is correspondingly small.

We therefore decided to make a new choice for $H$ : the three-generator subgroup $\langle b, a^{-1}ba, aba^{-1} \rangle$ (also of index 25 in $B_5/\gamma_5 B_5$ ) was selected. Given a presentation $P$ , steps (a)-(c) (with automatic selection of eliminations by the Tietze program) usually produce a presentation for $H = \langle b, a^{-1}ba, aba^{-1} \rangle$ on these generators and containing about 70 relators (including the fifth powers of the generators). Many of these relators will be redundant, as the 5-multiplier of $H$ should have rank 9. It may be possible to determine the order of $H$, using the shortest dozen or so relators, by enumerating the cosets of $\langle b \rangle$ (or $\langle a^{-1}ba \rangle$ or $\langle aba^{-1} \rangle$): the expected index is $5^5 = 3125$.

In this way all the presentations in the Table except the fifth have been shown to define $B_5/\gamma_5 B_5$. The calculations were performed by an implementation of the Felsch method of coset enumeration written by Havas and Alford. They were of varying degrees of difficulty, as shown in the Table by the total numbers of cosets defined. (The exact figures depend on the number of relators used and the subgroup whose cosets are enumerated; those shown in the Table are the smallest we have achieved in each case.)

Note that the (2, 6) presentations, involving the sets $S_2$ and $S_3$ of fifth powers, are optimal. It remains an open question whether either of the presentations

$$\langle a, b \mid [c_4, c_3], [c_5, c_3], S_j \rangle \quad (j = 1 \text{ or } 4)$$

defines $B_5/\gamma_5 B_5$. In the case of $S_1$, it was not possible to show that any of the three known order-five subgroups of $H$ had finite index, even when some 150000 cosets were defined. With $S_4$, we were unable to obtain a presentation for $H$ on fewer than four generators.

A comparison between the two approaches can now be made. The first (Tietze transformations in $G'$ ) depends heavily on the presence of commutator relators in $G'$ , and hence is more successful with presentations $P$ in which many of the $c_i$ commute. Indeed, this method failed in all cases (e.g. 4, 6 and 7 in the Table) where it is not evident from $P$ that at least one of the $c_i$ is central in $G$ . This provides little encouragement for the application of such an approach in an attack on $B_5$ itself, where only fifth-power relators would be available.

The second approach $\left(\text{coset enumeration in } \langle b, a^{-1}ba, aba^{-1}\rangle\right)$ is not so much affected by the form of the presentation $P$ , but is subject to the usual random vagaries of coset enumeration. Although generally more successful than the first approach, it was unable (with the available storage) to produce any (3, 5) presentation. In principle one can envisage applying enumeration-based techniques to a study of presentations of higher-class quotients of $B_5$ , but storage requirements may be expected to increase along with the subgroup indices to be determined.

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