Two Groups which Act on Cubic Graphs

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1 Introduction

Biggs [1] studies automorphism groups which act on cubic graphs. In particular he describes two families of groups:

$$\begin{aligned}4^+(a^l) = \langle a, b, \sigma \mid {} & \sigma^2 = (\sigma a)^2 = (\sigma b)^2 = (a^{-1}b)^2 = (a^{-2}b^2)^2 = 1, \\ & a^3b^{-3}a^3 = bab, a^3b\sigma a^4 = ba^2b, a^l = 1\rangle;\end{aligned}$$

$$\begin{aligned}5^+(a^l) = \langle a, b, \sigma \mid {} & \sigma^2 = (\sigma a)^2 = (\sigma b)^2 = (a^{-1}b)^2 = (a^{-2}b^2)^2 = (a^{-3}b^3)^2 = 1, \\ & a^4b^{-4}a^4 = ba^2b, a^4b\sigma a^5 = ba^3b, a^l = 1\rangle.\end{aligned}$$

In the tables presented in his talk at the Symposium, Biggs indicated that the orders of $5^+(a^{16})$ and $4^+(a^{12})$ were unknown but that the groups have homomorphic images with orders $2^{16}.30.48$ and $2^8.14.24$. He asked for more information about these groups. We show that both these groups have considerably larger quotients than those quoted above and present here details of computations done at the Symposium which prove this. We expect that these computations may form the basis of a proof that both these groups are infinite.

2 The Group $5^+(a^{16})$

In Biggs [1], it is shown that $G = 5^+(a^{16})$ has a quotient of order $2^{16}.30.48$, and the question of the finiteness of G is raised. Initial attempts to prove finiteness involved coset enumerations in G over the subgroups

$$K_1 = \langle a^{-1}b, a^{-2}b^2, a^{-3}b^3, a^{-4}b^4, a^{-5}b^5\rangle$$

of order 48 and

$$\begin{aligned}K_2 = \langle & a^8, b^{10}, (ab)^6, (a^2b)^8, (ab^2)^4, (a^2b^2)^5, (a^3b)^6, (ab^3)^4, (a^4b)^8, \\ & (a^3b^2)^4, (a^2b^3)^{10}, (ab^4)^8, (a^2b^4)^5, (a^3b^3)^6, (a^4b^2)^5, (a^5b)^6\rangle\end{aligned}$$

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of index expected to be a multiple of 1440. Both of these enumerations were unsuccessful, for reasons which will become obvious. These coset enumerations were performed with computer programs designed and implemented by Havas and Alford which are descendants of that described in [4].

Because of the difficulty of these coset enumerations, we decided to try to show that G was infinite, or at least very large. The approach taken was to find a subgroup which can be shown to have either an infinite or a very large quotient. There was no obvious subgroup to try, so we started by looking for possible candidates. There are two approaches which we used here. On the one hand Cayley (see [2]) was used to find low index subgroups directly from the presentation for G. On the other hand we looked for subgroups by using the random coincidence technique combined with coset enumeration [3].

Cayley (used by Cannon himself) provided a subgroup of index 90 with a concise generating set, namely

$$K_3 = \langle a, \sigma, b^2abab^2, ba^7b, baba^5bab \rangle.$$

The random coincidence technique found subgroups with index 144, 180, 240, 288, 360, 480 and 720 in the quotient of G obtained by adding the relation $b^{20} = 1$. These subgroups are all extensions of the image of K_2 in this quotient of G.

Since K_3 was derived with by far the neatest generating set of all these subgroups, we elected to study it. A presentation for K_3 was found using the Reidemeister-Schreier program described in [5]. This presentation on 181 generators with 504 relators, was used as input to the Tietze transformation program described in [7]. The program reduced the presentation for K_3 to the following 4-generator 22-relator presentation

$$\begin{aligned} K_3 = \langle a, b, c, d \mid {} & a^2, (ab)^2, (ac)^2, (ad^{-1})^2, (adc^{-1}d)^2, (bc^{-1})^4, \\ & bc^{-1}bd^2c^{-1}d^2, (cd^3)^2, (bc^{-1}d^{-1}bdc^{-1})^2, (bdb^{-1}d^3)^2, (bd^5)^2, \\ & (cd^2c^{-1}d^{-2})^2, (acb^{-1}d^{-5})^2, (ad^2b^{-1})^4, (bd^{-2}c^{-1}d^{-1}c^{-1}d^{-2})^2, \\ & d^{16}, (bc^{-1}d^{-1}bc^{-1}d^{-1}cdc^{-1}d)^2, (ab^{-1}cb^{-1}cd^2c^{-1}d^{-1}c^{-1}d)^2, \\ & (ad^3bc^{-1}d^{-1}bc^{-1}d^{-2})^2, (cdcdcd^{-1}c^{-1}d^{-1}cd^{-1}c^{-1}d^{-1})^2, \\ & (ad^{-1}cd^2cd^{-2}c^{-1}b)^4, (acdc^{-1}dbc^{-1}d^{-2}cd^{-1}adcb^{-1}d^{-1}cd^{-1})^4 \rangle. \end{aligned}$$

The abelian quotient of K_3 is elementary abelian of order 16. The maximal 2-class 13 quotient of K_3, obtained by the nilpotent quotient algorithm program (see [6]), has order 2^{130}. This shows that G has order which is a multiple of $90 . 2^{130}$.

One plausible approach, within this framework, to showing G infinite is to find a subgroup which has an infinite cyclic quotient. In an effort to do this, which was to prove unsuccessful, we obtained a presentation for $K_4 = K_3'$. The derived quotient of K_4 is elementary abelian of order 2^6 and K_4 has a maximal 2-class 6 quotient of order 2^{138}, showing that G has order at least $90 . 16 . 2^{138}$.

3 The Group $4^+(a^{12})$

Biggs showed that the group $H = 4^+(a^{12})$ has a quotient of order $2^8 . 14 . 24$ and asked whether H is finite. We studied H with the same overall techniques as applied to G. However we found the initial subgroup in a different way.

Consider the quotient L of H obtained by adding the relation $b^{16} = 1$. Then, by coset enumeration, the subgroup

$$\langle a^{-1}b, a^{-2}b^2, a^{-3}b^3, a^{-4}b^4 \rangle$$

has index 3584. Random coincidences in this coset table yield that

$$\langle a^{-1}b, a^{-2}b^2, a^{-3}b^3, a^{-4}b^4, bab^2ababa^2 \rangle$$

has index 14 in L. By coset enumeration we see also that

$$H_1 = \langle a^{-1}b, a^{-2}b^2, a^{-3}b^3, a^{-4}b^4, bab^2ababa^2 \rangle$$

has index 14 in H.

Using the Reidemeister-Schreier and Tietze programs, we obtain the following presentation for H_1:

$$\begin{aligned} \langle a,b,c,d \mid a^2 &= b^2 = c^2 = d^2 = (ab)^2 = (cd)^2 = (ac)^3 = (ad)^3 \\ &= (bc)^4 = abcbcacbc = (bdc)^4 = (adb)^6 = 1 \rangle. \end{aligned}$$

H_1/H_1' is cyclic of order 2 and, using the Reidemeister-Schreier and Tietze programs again, we obtain a presentation for $H_2 = H_1'$:

$$\begin{aligned} \langle a,b,c \mid a^2 &= b^3 = c^3 = (bc^{-1})^2 = (ab)^3 = (ab^{-1}c)^4 = (acb^{-1})^4 \\ &= (ac^{-1})^6 = 1 \rangle. \end{aligned}$$

Now $H_3 = H_2'$ has index 3 in H_2 and the same method as above yields

$$\begin{aligned} H_3 = \langle a,b,c,d \mid a^2 &= b^2 = c^2 = d^2 = (ac)^2 = (bd)^2 = (ab)^4 = (bc)^4 \\ &= (cd)^4 = (acd)^4 = (adb)^4 = (abcdacbd)^2 = (acdb)^4 = 1 \rangle. \end{aligned}$$

H_3/H_3' is elementary abelian of order 16 and the nilpotent quotient algorithm program shows that H_3 has a maximal 2-class 19 quotient of order 2^{76} (and in fact for $n < 20$ the maximal 2-class n quotient has order 2^{4n}). Thus H has order which is a multiple of $14 . 2 . 3 . 2^{76}$.

Continuing down the derived series of H_1, we obtained a presentation for $H_4 = H_3'$. Now H_4/H_4' is elementary abelian order 2^9 and H_4 has a maximal 2-class 9 quotient of order 2^{73} (for $n < 10$ the maximal 2-class n quotient has order 2^{8n+1}). This shows that H has order at least $14 . 2 . 3 . 16 . 2^{73}$.

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