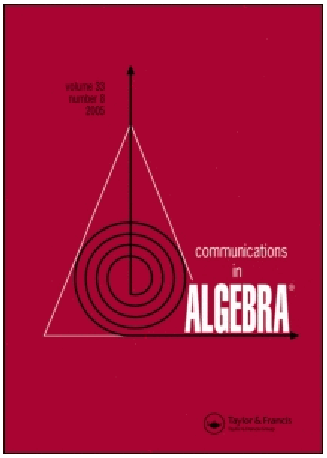

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Central factors of deficiency zero groups

George Havas [a]; Edmund F. Robertson [b]
[a] Department of Computer Science, The University of Queensland, Queensland, Australia
[b] Mathematical Institute, University of St. Andrews, Scotland

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CENTRAL FACTORS OF DEFICIENCY ZERO GROUPS

GEORGE HAVAS
Department of Computer Science, The University of Queensland,
Queensland 4072, Australia

EDMUND F. ROBERTSON
Mathematical Institute, University of St. Andrews,
KY16 9SS, Scotland

ABSTRACT

We answer a question of some twenty years standing: are the central factors of nilpotent groups of deficiency zero 3-generated? We prove that the answer is negative by giving an explicit presentation for a 3-generator, 3-relator group of order 2^{17} and class 5 which has central factors which are 4-generated but not 3-generated. We outline the computational techniques which lead to this result.

1 INTRODUCTION

Arising out of the work studied in [10], Wamsley informally asked the following question.

Are the central factors of nilpotent groups of deficiency zero 3-generated?

The question arose because all known examples of such deficiency zero groups had central factors which require no more than 3-generators. Furthermore, for any group G which is finite and of deficiency zero, G/G' is at most 3-generated [7, Theorem 9], leading to the thought that the same restriction might hold on central factors of nilpotent groups of deficiency zero in general. The question explicitly appears in [6, Question 1] and [8, Problem 8.12]. We prove that the central factors of nilpotent groups of deficiency zero are not necessarily 3-generated by giving an explicit presentation for a group which has central factors which are 4-generated but not 3-generated.

The group defined by a finite presentation $\{X : \mathcal{R}\}$ is well known to be infinite if $|X| > |\mathcal{R}|$. A group is said to have *deficiency zero* if it has a finite presentation $\{X : \mathcal{R}\}$ with $|X| = |\mathcal{R}|$ and $|Y| \leq |\mathcal{S}|$ for all other finite presentations $\{Y : \mathcal{S}\}$ of it. A recent computational investigation of groups of deficiency zero appears in [4], which includes much background material and some additional relevant references.

Havas, Newman and O'Brien [4] systematically study groups of deficiency zero, concentrating on groups of prime-power order. They prove that a number of p-groups have deficiency zero and give explicit balanced presentations for them. This significantly increases the number of such groups known. They describe a reasonably general computational approach which leads to their results. Like previous examples, none of theirs have central factors which require more than 3 generators. (This is not surprising, since they were concentrating on minimal groups, and a group with a 4-generator central factor can not be minimal.) We use the techniques of that paper to find our example.

We use the usual notation G', G'' for terms of the derived series of a group G and $G = G_1$, $G' = G_2$, G_3, ... for terms of the lower central series.

2 Our example

Let $G = \langle a, b, c \ : \ ab^{-1}c^{-1}bac,\ ba^{-1}c^{-1}ba^2ca^{-1},\ ac^2a^{-2}b^{-1}ab\rangle$. Then G is a group of order 2^{17} with class 5 and its lower central factors are elementary abelian with orders 2^3, 2^3, 2^4, 2^4 and 2^3. These facts are readily confirmed using standard tools for computational group theory, such as GAP [9], MAGMA [1] or stand-alone programs [5].

Once we have the presentation, the hard part of the computations is the coset enumeration required to prove the group finite. We used MAGMA for our initial calculations and checked them with other systems. The finiteness of G readily follows from any of the three enumerations over a cyclic subgroup generated by one of the group generators, and enumeration can in fact be completed over the trivial subgroup using moderate resources. With the coset enumerator in MAGMA, which is a descendant of the one described in [2], performance statistics are: index 4096 with a maximum of 88907 and total of 91916 cosets for the enumeration over $\langle a \rangle$; index 8192 with a maximum of 99397 and total of 102448 cosets for the enumeration over $\langle b \rangle$; index 4096 with a maximum of 61168 and total of 63044 cosets for the enumeration over $\langle c \rangle$; and index 131072 with a maximum of 712452 and total of 735861 cosets for the enumeration over the trivial subgroup. All of the enumerations reported in this paper were completed using the procedure defined by the MAGMA parameters `Strategy := <1000,1>` and `SubgroupRelations := -1`.

3 Finding our example

Using the methods described in [4], we searched a collection of presentations which might contain a group with a central factor of rank 4. Since we planned to prove finiteness by coset enumeration the easiest examples to handle would be 2-groups.

After trying some shorter presentations without success, we investigated presentations of length 22, using relators generated from the sequences:

$$\begin{aligned} b, b^{-1}, c, c^{-1}, a, a \quad &; \\ a, a^{-1}, a, a^{-1}, c, c^{-1}, b, b \quad &; \\ a, a^{-1}, a, a^{-1}, b, b^{-1}, c, c \quad &. \end{aligned}$$

Thus, we calculated the p-quotient determined by each presentation to an appropriate class; if this quotient is suitable, we calculated the largest metabelian quotient; if this also is suitable, we tried to prove that the group is finite by coset enumeration.

These presentations are of the type already studied by Havas, Newman and O'Brien, but they were looking for minimal p-groups of deficiency-zero. The presentations are convenient for our purposes because the squares of all generators are in the commutator subgroup, which implies that the lower exponent-2 central series used in ANU p-Quotient Program [3] coincides with the ordinary lower central series. For these groups the ranks of the first two central factors are both 3, so the smallest example G useful to us would have G_3/G_4 with rank 4.

We looked for presentations determining groups whose maximal class-3 quotients have order 2^{10} (so that G_3/G_4 has rank 4) and whose maximal nilpotent quotients actually have class at most 5. For such presentations we checked that the metabelian quotients were suitable by computing the abelian quotient invariants of their commutator subgroups.

There are eight presentations in this collection which passed these filters. For these the abelian quotient invariants of the commutator subgroup were either [2, 2, 2, 4, 8] or [2, 2, 2, 8, 8]. Coset enumerations over cyclic subgroups for seven of these presentations do not readily complete. The remaining presentation provides our example G, with $G'/G'' \cong C_2^3 \times C_8^2$.

For the record, these computations were first performed on a Toshiba T2450CT Personal Computer (75MHz DX4, 24MB RAM) using the PC version of MAGMA. The filtering took 750 cpu seconds. Initial coset enumerations for all eight presentations, restricted to a maximum of 50000 cosets, took 210 cpu seconds. None of these completed, but performance statistics showed more coincidences for our example group so we focussed our attention on it. The final coset enumerations for the example group over the three cyclic subgroups and the trivial subgroup took 1220 seconds.

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