

On the Automorphism Groups of Quasiprimitive Almost Simple Graphs

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Let Γ be a graph and let G be a subgroup of automorphisms of Γ . Then G is said to be *locally primitive* on Γ if, for each vertex v , the stabilizer G_v induces a primitive group of permutations on the set of vertices adjacent to v . This paper investigates pairs (Γ, G) for which G is locally primitive on Γ , G is an almost simple group (that is, $L \leq G \leq \text{Aut}(L)$ for some nonabelian simple group L), and the simple socle L is transitive on vertices. Each such graph is a cover of a possibly smaller graph $\tilde{\Gamma}$ on which G is also locally primitive, and for which in addition $\text{Aut } \tilde{\Gamma}$ is quasiprimitive on vertices (that is, every nontrivial normal subgroup of $\text{Aut } \tilde{\Gamma}$ is vertex-transitive). It is proved that $\text{Aut } \tilde{\Gamma}$ is also an almost simple group. In the general case in which $\text{Aut } \Gamma$ is not quasiprimitive on vertices, we show that either every intransitive minimal normal subgroup of $\text{Aut } \Gamma$ centralizes L , or L is of Lie type and $\text{Aut } \Gamma$ involves an explicitly known same characteristic module for L . © 1999 Academic Press

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1. INTRODUCTION

This paper investigates the automorphism groups of a class of finite arc-transitive graphs, namely finite graphs Γ which admit an almost simple group G of automorphisms such that G is locally-primitive on Γ , and the simple socle of G is vertex-transitive. The aim is to understand the structure of the full automorphism group $\text{Aut } \Gamma$ of Γ , given its almost simple subgroup G . Before stating our results we give definitions of these concepts, and we explain why these “almost simple graphs” are of special interest.

Some Definitions

A graph $\Gamma = (V, E)$ consists of a set V of vertices and a set E of unordered pairs from V , called *edges*. Automorphisms of Γ are permutations of V which leave E invariant. The set of all automorphisms of Γ is a subgroup of the symmetric group $\text{Sym}(V)$ of all permutations of V , and is called the *automorphism group* of Γ and is denoted $\text{Aut } \Gamma$. For $v \in V$, $\Gamma(v)$ denotes the set of neighbours of v , that is, the set of all vertices $u \in V$ such that $\{u, v\} \in E$. We say that Γ is a *cover* of a graph $\tilde{\Gamma} = (\tilde{V}, \tilde{E})$ if there is an edge-preserving surjection $\varphi: V \rightarrow \tilde{V}$ such that, for all $v \in V$, φ maps $\Gamma(v)$ bijectively onto $\tilde{\Gamma}(\varphi v)$.

A subgroup $G \leq \text{Sym}(V)$ is said to be *quasiprimitive* on V if each of its nontrivial normal subgroups is transitive on V ; and G is said to be *primitive* on V if the only G -invariant partitions of V are the trivial ones $\{V\}$ and $\{\{v\} \mid v \in V\}$. Since the orbits in V of a normal subgroup of G form a G -invariant partition of V , it follows that all primitive permutation groups are quasiprimitive (but the converse is not true). The *socle* of a group G is the product of its minimal normal subgroups and is denoted $\text{soc}(G)$. A group G is said to be *almost simple* if $L \leq G \leq \text{Aut}(L)$ for some nonabelian simple group L ; since L is the unique minimal normal subgroup, $L = \text{soc}(G)$.

Let $G \leq \text{Aut } \Gamma$. Then G is said to be *locally primitive* on Γ , and Γ is said to be *G -locally primitive* if, for each $v \in V$, the stabilizer G_v induces a primitive permutation group on $\Gamma(v)$ (and Γ is *locally primitive* if it is $\text{Aut } \Gamma$ -locally primitive). If $\Gamma = (V, E)$ is connected and G -locally primitive, then either (i) G is transitive on V and hence on the *arcs* of Γ (that is, the ordered pairs (u, v) for $\{u, v\} \in E$), or (ii) Γ is bipartite and G has two orbits in V , namely the two parts of the bipartition of V . Moreover, if Γ is bipartite and (i) holds, then G has a normal subgroup G^+ of index 2 such that Γ is G^+ -locally primitive, and (ii) holds for G^+ . The class of locally primitive graphs contains, as a proper subclass, the much studied class of 2-arc-transitive graphs. (A *2-arc* of a graph Γ is a triple (u, v, w) of vertices such that both $\{u, v\}$ and $\{v, w\}$ are edges, and $u \neq w$; and Γ is 2-arc-transitive if $\text{Aut } \Gamma$ is transitive on the set of 2-arcs of Γ .)

Some Background

The graphs we will study in this paper satisfy the following hypothesis.

HYPOTHESIS 1.1. *The graph $\Gamma = (V, E)$ is finite, connected, and G -locally primitive, where G is an almost simple subgroup of $\text{Aut } \Gamma$, and its socle $L := \text{soc}(G)$ is transitive on V .*

From the remarks above it follows that such a graph Γ is not bipartite and the group G is transitive on the arcs of Γ (for otherwise the simple group L would have a normal subgroup $L \cap G^+$ of index 2). Moreover, since L is the unique minimal normal subgroup of G , the group G is quasiprimitive on V .

Now we discuss why graphs satisfying Hypothesis 1.1 are of particular interest in a study of locally primitive graphs. Let $\Gamma = (V, E)$ be a finite connected G -locally primitive graph which is not bipartite, where $G \leq \text{Aut } \Gamma$. In [12] it was proved that Γ is a cover of a $\tilde{G}$ -locally primitive graph $\tilde{\Gamma} = (\tilde{V}, \tilde{E})$ such that $\tilde{G}$ is quasiprimitive on $\tilde{V}$ and $\tilde{G} = G/N$ for some intransitive normal subgroup N of G . (In fact $\tilde{V}$ is the set of N -orbits in V .) Thus to understand the structure of typical connected, nonbipartite, locally primitive graphs, it is important to understand the subclass consisting of those graphs Γ for which there exists $G \leq \text{Aut } \Gamma$ with G both locally primitive on Γ and quasiprimitive on vertices. Note in particular that, because of the relationship described above between the quasiprimitive members and the typical members of this class of graphs, we should not assume that the quasiprimitive, locally primitive group G is equal to $\text{Aut } \Gamma$.

The quasiprimitive subgroups of $\text{Sym}(V)$ have been subdivided into eight disjoint families (see [15] or the original classification in [14]). One of these families consists of the almost simple subgroups G of $\text{Sym}(V)$ such that $\text{soc}(G)$ is transitive on V . This family attracts special attention both because it provides many interesting families of examples (of permutation groups, of graphs, etc.) and because several problems in group theory and combinatorics can be reduced to the almost simple case (for example, classifying distance transitive graphs [16] and attempting to prove Weiss's conjecture for locally primitive graphs [4]).

In this paper we study the almost simple examples of quasiprimitive, locally primitive graphs, that is, pairs (Γ, G) such that Hypothesis 1.1 holds. Natural questions arise concerning the relationship between G and $\text{Aut } \Gamma$. Since G is locally primitive on Γ , $\text{Aut } \Gamma$ is automatically locally primitive on Γ , but, for example, $\text{Aut } \Gamma$ need not be quasiprimitive on vertices. This is a problem even for 2-arc transitive graphs (see [5, 6, 10]). Moreover, even if $\text{Aut } \Gamma$ is quasiprimitive on vertices, we are faced with the question of which of the eight types of quasiprimitive groups it might be.

The special case where $L = \text{soc}(G)$ is regular on V , that is, where $L_v = 1$ for $v \in V$, requires methods of analysis different from those needed for the general case where $L_v \neq 1$. This special case was considered in [8, Theorem 1.1], where it was shown that $\text{Aut } \Gamma$ is always almost simple and quasiprimitive on vertices, and its socle is equal to L .

Our investigation therefore focuses on the general case with $L_v \neq 1$. In this case, L is arc-transitive on Γ . As mentioned above, there are examples of pairs (Γ, G) satisfying Hypothesis 1.1 for which $\text{Aut } \Gamma$ is not quasiprimitive on vertices, but for all the known examples, $\text{Aut } \Gamma$ has a non-trivial normal subgroup which centralizes L . We addressed the question of whether there is any other situation in which $\text{Aut } \Gamma$ fails to be quasiprimitive on vertices. For many families of almost simple groups we can prove that the answer is no. The exceptions are certain families of Lie-type simple groups G where $\text{Aut } \Gamma$ may involve a known small same characteristic module for G . However, we have been unable to decide whether or not there exist graphs satisfying Hypothesis 1.1 in this exceptional case.

THEOREM 1.2. *Suppose that Hypothesis 1.1 holds for Γ , G , L , and that $\text{Aut } \Gamma$ is not quasiprimitive on V . Then there are two possible cases.*

(a) *Every intransitive minimal normal subgroup of $\text{Aut } \Gamma$ centralizes L .*

(b) *Alternatively: $L = S(q)$ is a simple group of Lie type over a field of order $q = p^e$, for some prime p ; there is an intransitive minimal normal subgroup of $\text{Aut } \Gamma$ which does not centralize L ; and each such subgroup N is an elementary abelian p -group. Moreover, if M is such that $C_N(L) < M \leq N$ and $\bar{M} = M/C_N(L)$ is a minimal nontrivial G -invariant subgroup of $N/C_N(L)$, then $\bar{M} = Z_p^d$, where e divides d , and L , d/e are as in one of the lines in Table I or II. If L , d/e appear in Table II, then N is the unique intransitive minimal normal subgroup of $\text{Aut } \Gamma$ not centralized by L .*

Consider the semidirect product $Y := N.G \leq \text{Aut } \Gamma$ in case (b) above. Here $C_N(L)$ is normal in Y , and hence by [12], Γ is a cover of a graph $\tilde{\Gamma}$ such that $\tilde{Y} := Y/C_N(L)$ is a subgroup of $\text{Aut } \tilde{\Gamma}$, and Hypothesis 1.1 holds for $\tilde{\Gamma}$ and $\tilde{G} := (C_N(L).G)/C_N(L) \cong G$. Thus in seeking examples in case

TABLE I

L	d/e	Comments on $\bar{M}$
$A_l^\pm(q)$	$l+1$	Natural module
$B_l(q)$	$2l+1$	Natural module
$C_l(q)$	$2l$	Natural module
$D_l^\pm(q)$	$2l$	Natural module

TABLE II

L	d/e	Comments on $\overline{M}$
$A_l^\pm(q), q \text{ odd}$	$l(l+1)/2$	
$B_l(q), l = 3 \text{ or } 4$	2^l	Spin module
$C_l(q), l = 3 \text{ or } 4$	2^l	Spin module
$D_l^\pm(q), l = 4 \text{ or } 5$	2^{l-1}	Spin module
${}^3D_4(q), q \text{ odd}$	12	
$E_6^\pm(q)$	27	
$E_7(q)$	56	

(b), we may assume that $C_N(L) = 1$, and hence that $M.G \leq \text{Aut } \Gamma$. We have not been successful as yet in constructing a connected graph satisfying all the conditions of Theorem 1.2(b). It would be very interesting to know if such graphs exist. Theorem 1.2 follows immediately from a more general result, Proposition 2.5, about arbitrary overgroups of G in $\text{Aut } \Gamma$. This will be proved in Section 2.

Suppose that (Γ, G) satisfy Hypothesis 1.1. If $\text{Aut } \Gamma$ is not quasiprimitive on V , and N is an intransitive normal subgroup of $\text{Aut } \Gamma$, then we must have $G \cap N = 1$, since the unique minimal normal subgroup L of G is transitive on V . It follows from [12] that Γ is a cover of a smaller graph $\tilde{\Gamma}$ which admits $\text{Aut } \Gamma/N$ as a locally primitive subgroup of automorphisms. Furthermore, Hypothesis 1.1 holds for $\tilde{\Gamma}$ and $\tilde{G} := NG/N \cong G$. Since G is finite, after a finite number of repetitions of this procedure we find that Γ is a cover of a graph Γ' such that (Γ', G) satisfies Hypothesis 1.1 with $\text{Aut } \Gamma'$ quasiprimitive on the vertices of Γ' . Thus it is important to study the case in which $\text{Aut } \Gamma$ is quasiprimitive on V . We show in this case that $\text{Aut } \Gamma$ must also be almost simple (possibly with socle different from L).

THEOREM 1.3. *Suppose that Hypothesis 1.1 holds for Γ, G, L , and that $\text{Aut } \Gamma$ is quasiprimitive on V . Then $\text{Aut } \Gamma$ is an almost simple group. Moreover, if Y is any quasiprimitive subgroup of $\text{Aut } \Gamma$ containing G , then either*

- (a) $\Gamma = K_8, G = \text{PSL}(2, 7), Y = \text{AGL}(3, 2)$; or
- (b) Y is an almost simple group.

Theorem 1.3 will be proved in Section 3. Note that the proofs of both Theorems 1.2 and 1.3 depend on the finite simple group classification.

2. THE PROOF OF THEOREM 1.2

First we state several lemmas which are used in the proof of our theorems. The first lemma follows immediately from [12] and the connectivity of Γ .

LEMMA 2.1. *Let Γ be a connected graph and suppose that Y is a subgroup of automorphisms of Γ such that Y is transitive on V and locally primitive on Γ . Let N be a normal subgroup of Y which is intransitive on V . Then either N is semiregular on V or Γ is bipartite with bipartite partition given by the two N -orbits on V .*

For a group G , let $m(G)$ denote the minimal index of a proper subgroup of G . The next lemma records an upper bound on the orders of Sylow subgroups of a simple group L in terms of $m(L)$.

LEMMA 2.2 [7, Lemma 2.1]. *Let L be a nonabelian simple group and let p be a prime divisor of $|L|$. Suppose that a Sylow p -subgroup of L has order p^a . Then $a \leq (m(L) - 1)/(p - 1)$ if $p \neq 2$, and $a \leq m(L) - 2$ if $p = 2$.*

The next lemma is used to study the situation where $\text{soc}(\text{Aut } \Gamma)$ is not abelian. It is an immediate consequence of the ‘‘Schreier Conjecture’’ (which is a consequence of the finite simple group classification) that the outer automorphism group of a finite nonabelian simple group is soluble.

LEMMA 2.3. *Let $Y = S : L$ be a semidirect product, where S and L are nonabelian simple groups. If L acts nontrivially by conjugation on S , then L is isomorphic to a subgroup of S , and in particular $|L| \leq |S|$.*

The next simple lemma is used to study the case where $\text{soc}(\text{Aut } \Gamma)$ is an elementary abelian p -group.

LEMMA 2.4. *Let L be a nonabelian simple group and let Y_0 be a semidirect product $N : L$ where $N = Z_p^m$ for some prime p and integer $m > 1$, and $Y_0 \not\cong N \times L$. Then L acts nontrivially by conjugation on $N/C_N(L)$.*

Proof. Write $C := C_N(L)$. If L acts trivially by conjugation on N/C , then $Cx^y = Cx$, for all $x \in N$, $y \in L$. On the other hand, since L does not centralize N there exists some $y \in L$ with $(|y|, p) = 1$ and $x \in N \setminus C$ satisfying $x^y \neq x$. (Recall that, for a nonabelian simple group L , the set of all elements of L with order coprime to p generates L .) It follows that $x^y = cx$, for some $c \in C$ with $|c| = p$, and hence $x = x^{y^{|y|}} = c^{|y|}x$, which means $c^{|y|} = 1$. This is impossible since $(p, |y|) = 1$. Thus L acts nontrivially by conjugation on N/C . ■

Now we prove the main result of this section. Theorem 1.2 is an immediate consequence of it. A permutation group L on a set V is called *semiregular* if $L_v = 1$ for all $v \in V$.

PROPOSITION 2.5. *Suppose that Hypothesis 1.1 holds for Γ , G , L , and that $G \leq Y \leq \text{Aut } \Gamma$. Then $C = C_{\text{Aut } \Gamma}(L)$ is semiregular and intransitive on V and*

one of the following holds:

- (a) Y is quasiprimitive on V ; or
- (b) Y is not quasiprimitive on V , and each intransitive minimal normal subgroup N of Y is contained in C ; or
- (c) $L = S(q)$ is a simple group of Lie type over a field of order $q = p^e$, and there exists an intransitive minimal normal subgroup of Y which is not contained in C . Let N be such a subgroup. Then N is an elementary abelian p -group. Furthermore, if $N \cap C < M \leq N$ is such that $M/(N \cap C)$ is a minimal nontrivial G -invariant subgroup of $N/(N \cap C)$, then $M/(N \cap C) = Z_p^d$, where e divides d , and L , d/e are as in one of the lines in Table I or II. If L , d/e appear in Table II, then N is the unique intransitive minimal normal subgroup of Y not contained in C .

Proof. As discussed in the Introduction, Γ is not bipartite, and G is quasiprimitive on V . If L is regular on V , then by [8, Theorem 1.1], $C = 1$, and $\text{Aut } \Gamma \leq \text{Aut}(L)$, so that (a) holds for Y . Thus we may assume that L is not regular on V . Now the centralizer in $\text{Sym}(V)$ of a transitive nonregular subgroup L is semiregular and intransitive, and hence C is semiregular and intransitive on V .

Suppose that Y is not quasiprimitive on V , so there exists an intransitive minimal normal subgroup N of Y . Now $N = T_1 \times \cdots \times T_m \cong S^m$, for some simple group S and $m \geq 1$. Set $W := NG$. Since L is transitive on V , it follows that $N \cap G = 1$ and hence that $W = N : G$ is a semidirect product. Now W is transitive and locally primitive on V , and by Lemma 2.1, N is semiregular on V . If all such subgroups N are contained in C , then part (b) holds. So we may assume that $N \not\leq C$.

We claim first that $S = Z_p$ for some prime p . Suppose to the contrary that S is a nonabelian simple group. If L normalizes each simple direct factor T_i of N , then, since $N \not\leq C$, L must act nontrivially by conjugation on T_i for some i . By Lemma 2.3, $|L| \leq |S|$, and hence $|L| \leq |N| < |V|$ (since N is intransitive and semiregular on V), which contradicts the fact that L is transitive on V . Hence L has an orbit of length l on the simple direct factors of N , for some $l > 1$. From the definition of $m(L)$ we have $m(L) \leq l \leq m$. Let p be a prime divisor of $|S|$. Then $|N|$ is divisible by $p^{m(L)}$. Since N is semiregular on V , $|V|$ is divisible by $p^{m(L)}$, and hence $|L|$ is also. On the other hand, by Lemma 2.2, $|L|$ is not divisible by $p^{m(L)-1}$, which is a contradiction. Hence $S = Z_p$ for some prime p .

Since $N \not\leq C$, L acts nontrivially, and hence faithfully on N , whence G is isomorphic to a subgroup of $\text{GL}(m, p)$. In fact we can say more. By Lemma 2.4, there is a minimal L -invariant subgroup M^* of $N/(N \cap C)$ such that L acts nontrivially by conjugation on M^* , and hence there is a minimal G -invariant subgroup M of $N/(N \cap C)$ with $M^* \leq M$. Now L acts

nontrivially and hence faithfully on M , so $M = Z_p^d$ for some $d > 1$ and G is isomorphic to an irreducible subgroup of $\text{GL}(d, p)$. Moreover, since N is semiregular on V and L is transitive on V , p^m ($m \geq d$) divides $|V|$, which divides $|L|$. Then the arguments given in the proofs of [7, Lemmas 2.6 and 2.7] show that L is neither an alternating group nor a sporadic simple group. Thus L is a simple group of Lie type, say $L = S(q)$ over a field of order q . Arguing as in the proof of [7, Lemma 2.8], we deduce that $q = p^e$ for some $e \geq 1$. Then arguing as in the proof of [7, Lemma 2.9], we find that e divides d and either $L, d/e$ are as one of the lines of Table I or Table II, or L is one of $A_l^\pm(q)$, ${}^3D_4(q)$, or $G_2(q)$ (q even), with $d/e = l(l+1)/2, 12$, or 6 , respectively. In the case of the groups $A_l^\pm(q)$, ${}^3D_4(q)$, and $G_2(q)$, the discussion in [7, Remarks 2.10] shows that q cannot be even and hence L is $A_l^\pm(q)$ or ${}^3D_4(q)$, as in Table II.

Suppose finally that $L, d/e$ are as in one of the lines of Table II. Suppose that there is a second intransitive minimal normal subgroup K of Y such that $K \not\leq C$. Then the argument of the previous paragraph shows that $K \cong Z_p^n$ and that there is a subgroup H such that $K \cap C < H \leq K$ and $H/(K \cap C) \cong Z_p^{d'}$ is a minimal nontrivial G -invariant subgroup of $K/(K \cap C)$; moreover, either $d' = d$ or $L, d'/e$ occurs in one of the lines of Table I. Set $P = \langle N, K \rangle \cong N \times K$. If P is intransitive on V then, by Lemma 2.1, P is semiregular on V , while if P is transitive on V then (as P abelian) P is regular on V (see [17, Proposition 4.4]). In either case $|P|$ divides $|V|$, which divides $|L|$. However, for each of the lines of Table II, p^{2d} does not divide $|V|$. Therefore $d' < d$ and $L, d'/e$ occur in one of the lines of Table I. In this case $p^{d+d'}$ is again larger than the p -part of $|L|$. Hence the group N is the unique intransitive minimal normal subgroup not contained in C . ■

The argument at the end of the proof also shows that, for case (c) with L in Table II, L has only one nontrivial composition factor in N and in the case where $L = A_l^\pm(q)$, $d/e = l(l+1)/2$, we must have $N \cap C = 1$ and $M = N$.

3. THE PROOF OF THEOREM 1.3

The purpose of this section is to prove Theorem 1.3. Since in part (a), $\text{Aut } \Gamma = S_8$, which is almost simple, it is sufficient to prove that (a) or (b) holds for a quasiprimitive overgroup Y of G in $\text{Aut } \Gamma$. Write $B := \text{soc}(Y)$. Since Y is quasiprimitive, $B = T_1 \times \cdots \times T_m$, where each $T_i \cong S$ for some simple group S and $m \geq 1$. The first quasiprimitive type defined in [15] is type HA . In that case $S = Z_p$ and $|V| = p^m$, for some prime p , and for all other types S is nonabelian.

In the four-step proof we use some of the information about the various types of finite quasiprimitive groups defined in [2] (or see [14] or [15]). There are seven types in which S is a nonabelian simple group. The first of these is the almost simple type, where we have $m = 1$, and in the other six types $m > 1$. There are five types (namely types HS , HC , SD , CD , and TW) in which $|V| = |S|^i$ for some i such that $m/2 \leq i \leq m$. In the final type PA , there is a Y -invariant partition of V with d^m parts for some $d \geq m(S)$.

Proof of Theorem 1.3. Step 1: The case where Y is of type HA . Suppose that $S \cong Z_p$ for some prime p , and $|V| = p^m$. By [9, Corollary 2], either L is 2-transitive on V or $|V| = 27$, and $L \cong \text{PSU}(4, 2)$. In the latter case $Y \leq \text{AGL}(3, 3)$, which has no subgroup isomorphic to L . Hence L is 2-transitive. Note that all quasiprimitive groups of type HA are primitive (see [15]). By [13, Proposition 6.1] the only inclusion $G < Y$, such that L is 2-transitive and Y is of type HA , is $G = L = \text{PSL}(2, 7) < Y = \text{AGL}(3, 2)$. Then $\Gamma = K_8$ and part (a) holds. We shall therefore assume from now on that Y is not of type HA , so S is a nonabelian simple group.

Step 2: Proof that Y is of type AS or PA . Assume that Y is not almost simple, so $m \geq 2$. We claim that L normalizes each of the T_i . Suppose to the contrary that L permutes $\{T_1, \dots, T_m\}$ nontrivially, and that one of the L -orbits in this action has length $n \geq 2$. Then $n \leq m$, and if Y has type HC (so that Y has two orbits of length $m/2$ on the T_i) we even have $n \leq m/2$. On the other hand, n is the index in L of a proper subgroup, so $n \geq m(L)$. Let p be a prime divisor of $|V|$ and note that $|V|$ is a proper divisor of $|L|$ since L is arc-transitive. Thus the p -part of $|L|$ is p^{a_p} , where $a_p \geq 1$ and, by Lemma 2.2, $a_p \leq m(L) - 1$. If the type of Y were HS , HC , SD , CD , or TW then $|V|$ would be divisible by $|S|^{m/2}$ and hence by 2^m , but this is not the case since $m \geq n \geq m(L) > a_2$. Thus Y has type PA and in this case there is a Y -invariant partition of V with d^m parts, for some $d > 1$; choosing p to divide d , we have that p^m divides $|V|$ and hence p^m divides $|L|$, so $m \leq a_p < m(L) \leq n \leq m$, which is a contradiction.

Hence L normalizes each of the T_i . Then in its faithful conjugation action on B the subgroup of automorphisms of B induced by L is contained in $\text{Aut}(S)^m$. Moreover, since the only insoluble composition factors of $(\text{Aut}(S))^m$ are isomorphic to S (by the ‘‘Schreier Conjecture’’ mentioned in Section 2), the subgroup of $\text{Aut}(B)$ induced by L is contained in $(\text{Inn}(S))^m$ and hence $L \subseteq B$. Since L is simple it follows that $|L|$ divides $|S|$. On the other hand, $|V|$ is a proper divisor of $|L|$. Hence $|V|$ is a proper divisor of $|S|$. Therefore the type is not HS , HC , SD , CD , or TW , and hence Y is quasiprimitive of type PA and in this case the subgroup L is contained in B .

Step 3: Candidates for Y of type PA . For type PA , $\prod_{i \neq j} T_i$ is not transitive on V for any j , and hence L projects nontrivially onto each T_i and so L is contained in a diagonal subgroup D of $B = S^m$, that is a subgroup

D of the form

$$D = \{(s, s^{\sigma_2}, \dots, s^{\sigma_m}) \mid s \in S\},$$

for some $\sigma_2, \dots, \sigma_m \in \text{Aut}(S)$. By [14], Y preserves a partition Σ of V such that its action on the parts of Σ is permutationally isomorphic to its product action on some Cartesian product Ω_0^m . From now on we shall identify Σ with Ω_0^m . Choose $\omega \in \Omega_0$ and set $R = S_\omega$. For each $i = 1, \dots, m$, define the subset

$$\Omega_i = \{(\omega_1, \omega_2, \dots, \omega_m) \in \Omega_0^m \mid \omega_i = \omega\}.$$

So $|\Omega_i| = |\Omega_0|^{m-1}$, and the set stabilizer B_{Ω_i} is

$$B_{\Omega_i} = \{(s_1, \dots, s_m) \mid \text{all } s_j \in S, s_i \in R\}.$$

For $\rho = (\omega, \omega, \dots, \omega) \in \Omega_0^m$, we have $B_\rho = \bigcap_{i=1}^m B_{\Omega_i} = R^m$. Since L is transitive on V , D is transitive on V and hence also on Ω_0^m . Now

$$D_{\Omega_i} = \{(s, s^{\sigma_2}, \dots, s^{\sigma_m}) \mid s^{\sigma_i} \in R\} = \{(s, s^{\sigma_2}, \dots, s^{\sigma_m}) \mid s \in R^{\sigma_i^{-1}}\},$$

where we write $\sigma_1 = 1$. Thus, for D to be transitive on Ω_0^m , we must have

$$|D_{\Omega_i} : D_{\Omega_i} \cap D_{\Omega_j}| = |\Omega_0|, \quad \text{for all } i \neq j,$$

and the subgroup $D_{\Omega_i} \cap D_{\Omega_j}$ is $\{(s, s^{\sigma_2}, \dots, s^{\sigma_m}) \mid s \in R^{\sigma_i^{-1}} \cap R^{\sigma_j^{-1}}\}$. It follows that S factorizes as

$$S = R^{\sigma_i^{-1}} R^{\sigma_j^{-1}}, \quad \text{for all } i \neq j,$$

that is, in the language of [1] we have a full factorization of S and a multiple factorization of S if $m \geq 3$. Moreover, if $m \geq 3$, then for distinct i, j, l ,

$$|D_{\Omega_i} \cap D_{\Omega_j} : D_{\Omega_i} \cap D_{\Omega_j} \cap D_{\Omega_l}| = |\Omega_0|,$$

which implies that

$$S = (R^{\sigma_i^{-1}} \cap R^{\sigma_j^{-1}}) R^{\sigma_k^{-1}},$$

that is to say (again in the language of [1]), we have a strong multiple factorization of S . Note that the factors $R^{\sigma_i^{-1}}$ are all isomorphic and are conjugate under elements of $\text{Aut}(S)$. By [1, Theorem 1.2] it follows that $m = 2$, and, moreover, by [1, Theorem 1.1], S, R , and $|\Omega_0|$ are given in one of the columns of Table III.

Step 4: Proof that Y is not of type PA . Since L is transitive on $\Sigma = \Omega_0^2$, we have $S \cong D = LD_\rho$ and $D_\rho \cong R$. We also have the factorization $L = L_{\Omega_1} L_{\Omega_2}$ of L with $|L : L_{\Omega_1}| = |L : L_{\Omega_2}| = |\Omega_0|$, and $|\Omega_0|^2$ divides $|L|$. A careful check of the tables in [11] shows that there are no

TABLE III

S	A_6	M_{12}	$\mathrm{Sp}_4(q), q = 2^r \geq 4$	$P\Omega_8^+(q), q \geq 2$
R	A_5	M_{11}	$\mathrm{Sp}_2(q^2) \cdot 2$	$\Omega_7(q)$
$ \Omega_0 $	6	12	$q^2(q^2 - 1)/2$	$2q^3(q^4 - 1)/(4, q^4 - 1)$
$ S / \Omega_0 ^2$	10	$2^2.3.5.11$	$4(q^2 + 1)$	$(4, q^4 - 1)q^6(q^6 - 1)(q^2 - 1)/4$

proper subgroups L of $D \cong S$ with these properties for any of the groups S in Table III. (Note that if $S = P\Omega_8^+(q)$ and $L \cong \Omega_7(q)$ then we would have a strong multiple factorization of S with respect to the three subgroups $\Omega_7(q)^{\sigma^i}$, $i = 0, 1, 2$, where σ is a triality automorphism [11, Table 4, line 1], but this was shown not to arise in [1, Theorem 1.1].) Hence $S = L$. Thus L and Ω_0 are given in Table III. If G normalized each simple direct factor T_i , then $T_1 : G$ would contain an intransitive normal subgroup T_1 . Since $T_1 : G$ is locally primitive on V , by Lemma 2.1, T_1 would be semiregular. This is impossible since $|T_1| = |L| > |V|$. Hence G interchanges the two simple direct factors of B . Moreover, since $L \subseteq B$ (and L is transitive on V so that $G = LG_\alpha$), it follows that $L \neq G$, and that G_α interchanges the two simple direct factors of B .

Since L is transitive on V , $|V|$ divides $|L|$. On the other hand, we have $B \cong L \times L$ and $B_\rho = R \times R$, for $\rho = (\omega, \omega) \in \Omega_0^2$ and α a vertex in the part ρ of the partition Ω_0^2 of V . By [14], B_α is a subdirect product of B_ρ (that is, B_α projects onto each of the two direct factors R of B_ρ). Note that

$$|V| = |B : B_\rho| \cdot |B_\rho : B_\alpha| = |L : R|^2 \cdot |B_\rho : B_\alpha|,$$

so $|B_\rho : B_\alpha|$ divides $|L|/|L : R|^2 = |R|/|L : R|$ (which is given in the last row of Table III).

In all cases the derived subgroup R' of R is either a nonabelian simple group of index at most 2 in R , or (in the case of $P\Omega_8^+(q)$ with q odd) $R = R' = \Omega_7(q)$ is nearly simple with center of order 2. In particular, $|R'|$ does not divide $|R|/|L : R|$, and hence $|R'|$ does not divide $|B_\rho : B_\alpha|$. It follows that B_α contains $K := R' \times R'$ as a normal subgroup and $|B_\rho : B_\alpha|$ divides $|R : R'|^2$ (which is 1 or 4).

Next we examine the action of Y_α and certain of its subgroups on the set $\Gamma(\alpha)$. We show first that K acts nontrivially on $\Gamma(\alpha)$. Suppose to the contrary that K fixes $\Gamma(\alpha)$ pointwise. Then $|\Gamma(\alpha)|$ divides $|B_\alpha : K|$, which divides 4. Since Γ is not a cycle this means that $|\Gamma(\alpha)| = 4$ and that the kernel of the action of B_α on $\Gamma(\alpha)$ is K . For $\beta \in \Gamma(\alpha)$, we have $K \subseteq B_\beta$, and it follows, since $B_\beta^{\Gamma(\beta)} \cong B_\alpha^{\Gamma(\alpha)}$ and is abelian, that K is the kernel of the action of B_β on $\Gamma(\beta)$. This implies that K is normalized by $\langle B_\alpha, B_\beta \rangle = B$, which is a contradiction. Hence $K^{\Gamma(\alpha)}$ is a nontrivial normal subgroup of the primitive group $Y_\alpha^{\Gamma(\alpha)}$, and so $K^{\Gamma(\alpha)}$ is transitive. Note that this implies

in particular that $|\Gamma(\alpha)| \geq m(R') \geq 5$, and therefore that $\Gamma(\alpha)$ is not contained in a part of the partition Ω_0^2 since these parts have size $|B_\rho : B_\alpha| \leq 4$.

If $\rho' \in \Omega_0^2$ is the part containing the vertex $\beta \in \Gamma(\alpha)$, then $\Gamma(\alpha) \cap \rho'$ is nonempty and is a block of imprimitivity for the action of Y_α on $\Gamma(\alpha)$. Since Y_α is primitive on $\Gamma(\alpha)$ and since $\Gamma(\alpha) \not\subseteq \rho'$, it follows that $\Gamma(\alpha) \cap \rho' = \{\beta\}$. Thus since K is transitive on $\Gamma(\alpha)$ it follows that $\Gamma(\alpha)$ consists of one vertex from each part in some K -orbit Δ on Ω_0^2 . Moreover, since $\Gamma(\alpha)$ is left invariant by Y_α , this K -orbit Δ is invariant under Y_α . Recall that G_α , and hence also Y_α , interchanges the two simple direct factors of B , and any element of Y_α which interchanges them is of the form $(g_1, g_2)\sigma$ where the $g_i \in \text{Aut}(T_i)$ and $\sigma : (\omega_1, \omega_2) \mapsto (\omega_2, \omega_1)$ for all $(\omega_1, \omega_2) \in \Omega_0^2$. It follows that $\Delta \subseteq \{(\omega_1, \omega_2) \mid \omega_i \in \Omega_0 \setminus \{\omega\}\}$, and the K -orbits in this set which are invariant under Y_α are all of the form $\Delta = \Delta_1 \times \Delta_2$, where Δ_1 and Δ_2 are orbits of R' in $\Omega_0 \setminus \{\omega\}$ of equal size. Thus $|\Gamma(\alpha)| = |\Delta| = d_0^2 \geq 5$, where $d_0 = |\Delta_1| = |\Delta_2|$ is the length of an R' -orbit in $\Omega_0 \setminus \{\omega\}$.

Now we exploit the fact that $|\Gamma(\alpha)| = d_0^2$ divides $|L_\alpha|$, which divides $|L_\rho| = |L|/|\Omega_0|^2$, and $|L|/|\Omega_0|^2$ is listed in the last row of Table III. Thus L is not A_6 or M_{12} , as there are no possibilities for a square $d_0^2 \geq 5$ dividing $|L|/|\Omega_0|^2$ in these cases. Suppose next that $L = Sp_4(q)$ ($q = 2^a \geq 4$). Then $R' = Sp_2(q^2) \cong SL_2(q^2)$ and the minimal index of a proper subgroup is $m(R') = q^2 + 1$, so $d_0 \geq q^2 + 1$. This leads to a contradiction since in this case $|L|/|\Omega_0|^2$ is less than $(q^2 + 1)^2$. Hence $L = P\Omega_8^+(q)$ ($q \geq 2$), and $R = R' = \Omega_7(q)$. Here R is the stabilizer in L of a nonsingular 1-space $\omega \in \Omega_0$, and the length d_0 of an R -orbit in $\Omega_0 \setminus \{\omega\}$ is the index $|R : R_{\omega'}|$, for some nonsingular 1-space $\omega' \neq \omega$. Since $R_{\omega'}$ stabilizes the two-dimensional subspace U generated by ω and ω' , d_0 is divisible by $|R : R_U|$. Now U is not totally singular. If U were nonsingular then R_U would involve either $\Omega_6^+(q)$ or $\Omega_6^-(q)$ and $|R : R_U|$ would be divisible by $q^3 + 1$ or $q^3 - 1$, respectively. However neither $(q^3 + 1)^2$ nor $(q^3 - 1)^2$ divides $|L|/|\Omega_0|^2$. Hence U has a one-dimensional radical. However, in this case also, $|R : R_U|$ is divisible by $q^3 + 1$ and we again have a contradiction. This completes the proof.

4. COMPUTATIONAL REMARKS

Our proofs of the theorems in this paper are now essentially theoretical. However, it would be remiss of us to omit any mention of computations which helped lead us to the theorems and proofs. Thus we used MAGMA [3] to explicitly compute various graphs and their automorphism groups.

For the candidate groups in Table III, our computations revealed that none of the small cases in that list actually worked. This encouraged us to believe that the case PA did not occur in Theorem 1.3, as is now proved. As an example, we readily computed the A_6 -arc-transitive graphs and their

full automorphism groups with point stabilizer isomorphic to Z_5 or D_{10} , respectively. The computations showed that $|\text{Aut}\Gamma| < |A_6 \times A_6|$ and hence that the A_6 case did not arise.

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