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A Family of Non-quasiprimitive Graphs Admitting a Quasiprimitive 2-arc Transitive Group Action

XIN GUI FANG, GEORGE HAVAS AND JIE WANG

Let Γ be a simple graph and let G be a group of automorphisms of Γ. The graph is $(G, 2)$-*arc transitive* if G is transitive on the set of the 2-arcs of Γ. In this paper we construct a new family of $(\mathrm{PSU}(3, q^2), 2)$-arc transitive graphs Γ of valency 9 such that $\mathrm{Aut}\Gamma = Z_3.G$, for some almost simple group G with socle $\mathrm{PSU}(3, q^2)$. This gives a new infinite family of non-quasiprimitive almost simple graphs.

1. INTRODUCTION

A permutation group on a set Ω is said to be *quasiprimitive* if each of its non-trivial normal subgroups is transitive on Ω. A quasiprimitive group G is said to be of type AS if $S \leq G \leq \mathrm{Aut}(S)$, for some non-abelian simple group. Let Γ be a finite simple undirected graph with vertex set $V\Gamma$ and edge set $E\Gamma$ and let G be a group of automorphisms of Γ. The graph Γ is said to be G-*quasiprimitive* if and only if G is quasiprimitive on $V\Gamma$. In particular, an $\mathrm{Aut}\Gamma$-quasiprimitive graph is said to be *quasiprimitive*. A 2-arc of Γ is a triple $(\alpha_0, \alpha_1, \alpha_2)$ of vertices such that $\{\alpha_0, \alpha_1\}, \{\alpha_1, \alpha_2\} \in E\Gamma$ and $\alpha_0 \neq \alpha_2$. If G is transitive on the 2-arcs of Γ then Γ is said to be $(G, 2)$-*arc transitive*.

In [25] it is shown that every non-bipartite 2-arc transitive graph is a cover of a quasiprimitive graph. Thus, to classify 2-arc transitive graphs, a fundamental problem is classifying quasiprimitive 2-arc transitive graphs. For a G-quasiprimitive graph Γ, Praeger [26] raised the question: Under what conditions can we be certain that $\mathrm{Aut}\Gamma$ is quasiprimitive on $V\Gamma$? Baddeley [3, Section 6] constructed a connected $(G, 2)$-arc transitive graph Γ, for some quasiprimitive group G of twisted wreath type, and proved that its full automorphism group is isomorphic to $C \times G$ with C some small non-trivial cyclic group, which gives the first example where G is quasiprimitive but $\mathrm{Aut}\Gamma$ is non-quasiprimitive. He comments there that such graphs seem difficult to construct. In [20], Li found an infinite family of $(\mathrm{PSL}(2, q), 2)$-arc transitive graphs with valency 4 and proved their full automorphism groups to be isomorphic to $Z_2 \times \mathrm{PSL}(2, q)$. This gives the first infinite family of non-quasiprimitive graphs Γ such that $\mathrm{PSL}(2, q)$ acts quasiprimitively on $V\Gamma$. However, the approach in [20] is hard to generalize to larger valency cases, because valency 4 implies that the point stabilizer of the automorphism group is a $\{2, 3\}$-group.

Now we consider connected $(S, 2)$-arc transitive graphs for non-abelian simple groups S. Set $C := \mathrm{C}_{\mathrm{Aut}\Gamma}(S)$ and let Γ^* be the quotient graph of Γ modulo C-orbits on $V\Gamma$ (obtained by taking C-orbits as vertices and joining two C-orbits by an edge if there is at least one edge in Γ joining a point in the first C-orbit to a point in the second one). By [24], Γ is a cover of Γ^*. For $\alpha \in V\Gamma$, let $\Gamma(\alpha)$ denote the set of all vertices adjacent to α. Clearly, the point stabilizer S_α induces a permutation group on $\Gamma(\alpha)$, which is denoted by $S_\alpha^{\Gamma(\alpha)}$. Since S is transitive on the 2-arcs of Γ, the stabilizer S_α is 2-transitive on $\Gamma(\alpha)$ and so $S_\alpha^{\Gamma(\alpha)}$ is primitive. In [8, 10], Praeger and coworkers developed a general and systematic method to determine the automorphism group of certain types of symmetric graphs. Let $G = \mathrm{Aut}\Gamma \cap \mathrm{Aut}(S)$, then by [8, Theorems 1.1 and 1.2], we conclude immediately the following.

THEOREM 1.1. *Let Γ be a connected $(S, 2)$-arc transitive graph with S a finite alternating group, a sporadic simple group or one of the following simple groups of Lie type:* $\mathrm{PSL}(2, q)$,

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$\mathrm{U}_3(q)$, $G_2(q)$, $F_4(q)$, ${}^2B_2(q)$, ${}^2F_4(q)$ or ${}^3D_4(q)$ (q even for ${}^3D_4(q)$). If C is a cyclic group with prime order and if there is no quasiprimitive group of type AS in $\mathrm{Aut}\Gamma$ *containing $C.G$ as a maximal subgroup, then Γ is non-quasiprimitive. Further, if* $\mathrm{Aut}\Gamma^*$ *is quasiprimitive with socle S, then* $\mathrm{Aut}\Gamma = C.G$.

This theorem highlights a way to construct graphs where G is quasiprimitive but $\mathrm{Aut}\Gamma$ is non-quasiprimitive. In this paper, we first construct an infinite family of connected $(\mathrm{U}_3(q), 2)$-arc transitive graphs Γ of valency 9 such that the centralizer of $\mathrm{U}_3(q)$ in $\mathrm{Aut}\Gamma$ is Z_3. Then, using Theorem 1.1, we show that Γ is not quasiprimitive. Finally we prove that $\mathrm{Aut}\Gamma = Z_3.G$, where $G = \mathrm{Aut}(\mathrm{U}_3(q)) \cap \mathrm{Aut}\Gamma$. Thus we obtain a new infinite family of connected, non-quasiprimitive graphs Γ such that $\mathrm{U}_3(q)$ acts quasiprimitively on $V\Gamma$ and transitively on the 2-arcs of Γ. The main result of this paper is the following. Recall that $C = \mathrm{C}_{\mathrm{Aut}\Gamma}(S)$ and $G = \mathrm{Aut}(S) \cap \mathrm{Aut}\Gamma$.

THEOREM 1.2. *Let $S = \mathrm{U}_3(q)$, where $q = 2^m$ with $m = 3r$ for some odd integer r. Then there is a family of connected $(S, 2)$-arc transitive graphs Γ with valency 9 (see Construction (2.4)) which have non-quasiprimitive full automorphism groups. Further,* $\mathrm{Aut}\Gamma \cong Z_3 \times G$.

In Section 2 we construct an infinite family of connected $(\mathrm{U}_3(q), 2)$-arc transitive graphs with valency 9. Theorem 1.2 is proved in Section 3.

2. $(\mathrm{U}_3(q), 2)$-ARC TRANSITIVE GRAPHS WITH VALENCY 9

For a $(G, 2)$-arc transitive graph Γ, as described in [9, Section 2], if T is a core-free subgroup of G (that is $\cap_{x \in G} T^x = 1$) and g is a 2-element of G such that $g^2 \in T$, $\langle T, g\rangle = G$ and T is 2-transitive on the coset space $[T : T \cap T^g]$, then the graph $\Gamma' = \Gamma(G, T, TgT)$ with $V\Gamma' = [G : T]$ and $E\Gamma' = \{\{Tx, Ty\} \mid xy^{-1} \in TgT\}$ is a connected $(G, 2)$-arc transitive graph of valency $d := |T : T \cap T^g|$. For $T \in V\Gamma' = [G : T]$, write $\alpha = T$. Then $G_\alpha = T$ since the action of G on $V\Gamma'$ is by right multiplication. Conversely (see [9, Theorem 2.1]) each connected $(G, 2)$-arc transitive graph is isomorphic to $\Gamma(G, T, TgT)$ for some core-free subgroup T and 2-element g satisfying these properties. Note that all connected $(G, 2)$-arc transitive graphs constructed in this section are given in terms of G_α and g as above.

Now we give some information about $S := \mathrm{U}_3(q)$ for $q = 2^m$. Write $d = (3, q + 1)$. The order of S is $q^3(q^2 - 1)(q^3 + 1)/d$. Let τ be an automorphism of $\mathrm{GF}(q^2)$ with $\tau^2 = 1$. Write $d = (3, q + 1)$. For $k \in \mathrm{GF}(q^2)^\times$ and $a, b \in \mathrm{GF}(q^2)$ satisfying $a^{1+\tau}, b + b^\tau \in \mathrm{GF}(q)$ and $a^{1+\tau} + b + b^\tau = 0$, define

$$k_\tau := \begin{pmatrix} k^{-\tau} & 0 & 0 \\ 0 & k^{\tau-1} & 0 \\ 0 & 0 & k \end{pmatrix}, \qquad Q(a, b) := \begin{pmatrix} 1 & a & b \\ 0 & 1 & a^\tau \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $Q(q)$ be the subgroup of S generated by all the $Q(a, b)$ with a, b satisfying the above condition. Then, by [14, Chapter II, 10.12], $Q(q)$ is a Sylow 2-subgroup of S with order q^3 and $\mathrm{N}_S(Q(q)) = Q(q){:}K(q)$, where $K(q) = \{k_\tau \mid k \in \mathrm{GF}(q^2)^\times\} \cong Z_{(q^2-1)/d}$. For convenience we write $k_\tau = k$, $K(q) = K$, $Q(q) = Q$ and $H(q) = Q{:}K$. In particular, for a subfield $\mathrm{GF}(q')$ of $\mathrm{GF}(q)$, we write $K(q')$, $Q(q')$ and $H(q')$ respectively for the subgroups of K, Q and $H(q)$ with entries in $\mathrm{GF}(q')$. Let K_{q-1} and $K_{(q+1)/d}$ denote the subgroups of K with orders $q - 1$ and $(q + 1)/d$, respectively. The following two lemmas give some properties of S.

LEMMA 2.1 ([12]). *Let M be a maximal subgroup of S. Then either M is conjugate to one of the subgroups P_i given in the table below, or M is conjugate to $P_5(l) := \mathrm{U}_3(q_0)$ where $q_0 = 2^l$ for some divisor $l > 1$ of m such that m/l is prime.*

Group	*Structure*	*Remarks*
P_1	$Q{:}K$	the normalizer of Q in S
P_2	$[(q+1)/d].\mathrm{PSL}(2,q)$	*the stabilizer of a non-isotropic point*
P_3	$(Z_{q+1}\times Z_{(q+1)/d}){:}S_3$	the stabilizer of a unitary base
P_4	$Z_{(q^3+1)/(d,(q+1))}{:}3$	the type of $\mathrm{U}_1(q^3)$
P_6	$Z_3^2{:}Q_8$	$m=3r$ for some prime $r>3$
P_7	$Z_3^2{:}Q_8{:}$	$q=8$

LEMMA 2.2 ([14, II, 10.12]AND [15, VII SECTION 7, XI 13.9]). *The following properties hold in Q and $Q{:}K$.*
(a) *Q is disjoint from its conjugates, and $Z(Q)=Q'=\{Q(0,b)\mid b\in\mathrm{GF}(q^2),\ b+b^\tau=0\}$ is elementary abelian of order q and, for each $x\in Q\backslash Z(Q)$, x has order* 4. *Further, for any $Q(a,b)$, $Q(a',b')\in Q$,*

$$Q(a,b)Q(a',b')=Q(a+a',b+b'+aa'^\tau). \tag{1}$$

Moreover, each elementary abelian 2*-subgroup of S is conjugate to a subgroup of $Z(Q)$.*
(b) *K is cyclic of order $(q^2-1)/d$ and, for any $Q(a,b)\in Q$ and $k\in K$,*

$$k^{-1}Q(a,b)k=Q(k^{2\tau-1}a,k^{\tau+1}b). \tag{2}$$

Further K_{q-1} is regular on $Z(Q)\backslash\{1\}$ by conjugation, while $K_{(q+1)/d}$ is the centre of $Z(Q){:}K$.

The following proposition forms the basis of our constructions of connected $(S,2)$-arc transitive graphs with valency 9. Let $T=Z_3^2{:}Q_8<P_7<\mathrm{U}_3(8)$ with $Q_8<Q$ and set

$$\Delta(S,Q_8):=\{Q(0,\xi)\notin T\mid \mathbb{F}_2(\xi)=\mathrm{GF}(q)\}. \tag{3}$$

PROPOSITION 2.3. *Let $S=\mathrm{U}_3(q)$ with $q=2^m$, where $m=3r$ and r is odd. Let $T=Z_3^2{:}Q_8<P_7<\mathrm{U}_3(8)$ with $Q_8<Q$. For each $g\in\Delta(S,Q_8)$ (see* (3) *above), define $\Gamma=\Gamma(S,T,TgT)$. Then Γ is a connected $(S,2)$-arc transitive graph of valency* 9. *Moreover,* $\mathrm{Aut}\Gamma$ *contains a subgroup isomorphic to $Z_3\times S$.*

PROOF. By Lemma 2.2, all elements of S with order 2 are conjugate in $\mathrm{Aut}(S)$. So we assume that $Q(0,1)\in Q_8<T$. Since $Q_8<Q$ and $g\in\Delta(S,Q_8)\subset Z(Q)$, $Q_8\le T\cap T^g$. Since Q_8 is transitive by conjugation on $Z_3^2\backslash\{1\}$, $T\cap T^g$ is either T or Q_8. In the former case we deduce that a Sylow 2-subgroup of $\mathrm{N}_S(Z_3^2)$ has order at least 16. By Lemma 2.1, $\mathrm{N}_S(Z_3^2)=P_5(l)$, for some divisor l of m with $l>1$, which is impossible since $P_5(l)$ is simple. So $T\cap T^g=Q_8$. Thus Γ is a $(S,2)$-arc transitive graph with valency $d_\Gamma=|T:T\cap T^g|=9$. Now we show that $\langle T,g\rangle=S$. By Lemma 2.1 we have $\langle T,g\rangle=\mathrm{U}_3(q_1)^x$, for some $q_1=2^n$ with n a divisor of m and $x\in S$. Write $Y=\langle T,g\rangle$ and let Q_1 be the Sylow 2-subgroup of Y containing $\langle Q_8,g\rangle$. Since $Q_8<Q$, $Q_1\le Q$ by Lemma 2.2. Set $N:=\mathrm{N}_Y(Q_1)$. Then there exists some $\upsilon\in Q$ such that $N=Q_1{:}K_1^\upsilon$ with $|K_1|=(q_1^2-1)/d\le|K|$. Since $g\in Z(Q)$ and $\upsilon\in Q$, g centralizes υ^{-1}. Thus, replacing T and N by $T^{\upsilon^{-1}}$ and $N^{\upsilon^{-1}}$, respectively, we may assume further that $N=Q_1{:}K_1$, where $Q_1\cong Q(q_1)$ and $Q(0,1)\in Q_1$. Note that K contains a unique subgroup $K(q_1)$ of order $(q_1^2-1)/d$. Thus $K_1=K(q_1)$. Moreover, since Q_1 contains $Q(0,1)$ and since $K(q_1)$ acts transitively by conjugation on $Z(Q_1)\backslash\{1\}$, we have $Z(Q_1)=Z(Q(q_1))=\{Q(0,\xi)\mid\xi\in\mathrm{GF}(q_1)\}$. Now $g=Q(0,\xi)\in Z(Q_1)$ with $\mathbb{F}_2(\xi)=\mathrm{GF}(q)$, which implies that $q_1=q$, that is, $\langle T,g\rangle=S$. So Γ is connected.

Finally we show that $\mathrm{Aut}\Gamma$ contains a subgroup isomorphic to $Z_3\times S$. Now $\mathrm{N}_S(T)=T{:}B=P_7$ with $B=\langle u\rangle\cong Z_3$. By the Frattini argument we can choose B such that $Q_8\lhd Q_8{:}B$ and

$Z(Q_8) = \langle Q(0,1)\rangle$ with $|Q(0,1)| = 2$, $Q(0,1)^u = Q(0,1)$. By Lemma 2.2 (b), $u = k_1^v$, for some $k_1 \in K_{(q+1)/d}$ and $v \in Q$. For each $z \in Z(Q)$, again by Lemma 2.2 (b), $z^u = z$. In particular, $g^u = g$ since $g \in Z(Q)$. Thus $g^u \in TgT$. Then by [9, Proposition 2.4], $\mathrm{C}_{\mathrm{Aut}\Gamma}(S) = \langle\sigma_u\rangle \cong Z_3$ where $\sigma_u : Tz \mapsto Tuz$, for each $Tz \in V\Gamma$. Consequently $\mathrm{Aut}\Gamma$ contains a subgroup $\langle\sigma_u\rangle \times S \cong Z_3 \times S$. □

From Proposition 2.3 we obtain immediately an infinite family of $(\mathrm{U}_3(q), 2)$-arc transitive graphs with valency 9 defined as in the following.

CONSTRUCTION 2.4. *Let $S = U_3(q)$ with $q = 2^m$ and $m = 3r$ with r odd. Let $T = Z_3^2{:}Q_8 < P_7 < U_3(8)$ with $Q_8 < Q$. Define*

$$\Gamma(\mathrm{U}_3(q), T) := \{\Gamma(S, T, TgT) \mid g \in \Delta(S, Q_8)\},$$

where $\Delta(S, Q_8) = \{Q(0,\xi) \notin T \mid \mathbb{F}_2(\xi) = \mathrm{GF}(q)\}$.

3. PROOF OF THEOREM 1.2

In this section we complete the proof of Theorem 1.2. The first proposition shows that there is no quasiprimitive group of type AS containing $C.G$ as its maximal subgroup. Given groups A and B, let $A \circ B$ denote the central product of A and B.

PROPOSITION 3.1. *Let Y be an almost simple group such that $T \trianglelefteq Y \le Aut(T)$, where T is simple. Suppose that Y has a subgroup $M = 3.G$, where $\mathrm{U}_3(q) \trianglelefteq G \le \mathrm{Aut}(\mathrm{U}_3(q))$ for $q = 2^f > 2$. Then M is not maximal in Y.*

PROOF. Suppose otherwise that $M = 3.G$ is a maximal subgroup of Y. Thus M contains a normal subgroup of order 3, denoted by Z_3. Then the maximality of M implies that $M = \mathrm{N}_Y(Z_3)$. In other words, M is a maximal 3-local subgroup of Y. Note that M is unsolvable and has exactly one non-abelian composition factor $\mathrm{U}_3(q)$. If T is a sporadic simple group then, by checking the ATLAS [6], we can see that Y has no such maximal subgroup.

Next we suppose that T is a finite classical simple group defined over a field of characteristic r. Moreover, assume that Y contains no graph automorphism of order 3 when $T = \mathrm{P}\Omega^+(8, r^s)$. Then M must belong to one of the eight subgroup collections $\mathcal{C}_i$ $(1 \le i \le 8)$ defined by Aschbacher [1] (see also [19] for details). If T is a linear group $\mathrm{PSL}(n, r^s)$, a symplectic group $\mathrm{Sp}_n(r^s)$ or an orthogonal group $\mathrm{O}_n(r^s)$, then a case-by-case check shows that no such maximal subgroup exists (cf. [19, Chapter 4]).

If $T = \mathrm{U}_n(r^s)$ then, by [19, 4.1.4], $T = \mathrm{U}_4(2^f)$ and $M \cap T = [a].\mathrm{U}_3(2^f).[b]$ where a and b are positives given by formula 4.1.3 of [19]. In this case, since $G \le \mathrm{Aut}(\mathrm{U}_3(2^f))$, we have $3|\mathrm{U}_3(2^f)| \le |M| \le |\mathrm{U}_3(2^f)| \cdot 6f(2^f+1,3)$. On the other hand, by [19, 4.1.4], $|M \cap T| = (2^f+1)(2^f+1,3)|\mathrm{U}_3(2^f)|$, which implies $f \le 4$. If $f = 4$ or 2 then M is not a 3-local subgroup. If $f = 3$, by [19, formula 4.1.3], $M \cap T = [9].\mathrm{U}_3(8).3$, which is not of the form $3.G$, a contradiction.

Now we consider the case that $T = \mathrm{P}\Omega^+(8, r^s)$ and Y contains a graph automorphism of order 3. In this case, all maximal subgroups of Y are determined by Kleidman [17]. It follows that, if M is a local maximal subgroup of Y and has exactly one non-abelian composition factor $\mathrm{U}_3(q)$, then $M \ge (Z_{q+1} \times \mathrm{GU}_3(q)).2$ (see [17, 3.2.2]), which is not of the form $3.G$.

Now we consider the cases where T is an exceptional simple group of Lie type.

If $T = {}^3D_4(r^s)$ then, by [18], the only possibility for M is $2 < q = 2^f \equiv -1 \pmod 3$ and $M \cap T = (Z_{q^2-q+1} \circ \mathrm{SU}_3(q)).[6]$, which is not of the form $3.G$.

If $T = {}^2B_2(2^s)$ then, by [27], all maximal subgroups of Y are solvable except for ${}^2B_2(2^t)$ for some divisor t of s such that s/t is prime. Therefore, no maximal subgroup of form $3.G$ occurs in this case.

Finally assume that $T = C_2$, F_4, E_i $(i = 6, 7, 8)$, ${}^2G_2$, ${}^2F_4$ or ${}^2E_6$ defined over a finite field of characteristic r. Write $\mathrm{Inndiag}(T)$ as the group generated by all inner and diagonal automorphisms of T. It is obvious that $M = \mathrm{N}_Y(Z_3)$ and, by [5], there are four cases to be considered.

(1) M is a maximal parabolic subgroup of Y and $r = 3$.

(2) $Z_3 \not\leq \mathrm{Inndiag}(T)$.

(3) $Z_3 \leq \mathrm{Inndiag}(T)$ and M is of maximal rank (see [5] and [23] for the definition).

(4) $Z_3 \leq \mathrm{Inndiag}(T)$ and M is not of maximal rank.

If (1) holds, then M is an extension of a 3-group by the Chevalley group determined by a maximal subdiagram of the Dynkin diagram of T. It follows that no such maximal subgroup contains $\mathrm{U}_3(q)$ as a non-abelian composition factor.

If (2) holds then, by [11, Sections 7 and 9], $M \cap T = \mathrm{C}_T(\alpha)$ for an outer automorphism α of prime order. By [2, Section 19], [11, Section 9] and [5, Proposition 2.7], none of them contain $\mathrm{U}_3(q)$ as a non-abelian composition factor.

If (3) holds then all M are determined and listed in [23, Tables 5.1 and 5.2]. Among these maximal subgroups, no local one contains exactly one $\mathrm{U}_3(q)$ as its non-abelian composition factor.

If (4) holds then all possible M are listed in [5, Table 1]. None of them contain $\mathrm{U}_3(q)$ as its non-abelian composition factor. This completes the proof of the proposition. □

Let S, T, Γ be as in Construction 2.4. For any $Tz \in V\Gamma$, denote $\sigma_u : Tz \mapsto Tuz$ as in the proof of Proposition 2.3. Let Γ^* be the quotient graph of Γ modulo C-orbits, where $C\langle\sigma_u\rangle \cong Z_3$. Write $X := \mathrm{Aut}\Gamma$, and $X^* := \mathrm{Aut}\Gamma^*$. The next proposition shows that the socle of X^* is S.

PROPOSITION 3.2. *Let* Γ *and* $S = \mathrm{U}_3(q)$ *be given as in Construction* 2.4 *and let* Γ^* *be the quotient graph of* Γ *modulo* C*-orbits on* $V\Gamma$*. Then* X^* *is quasiprimitive with socle isomorphic to* S.

PROOF. By [24], Γ is a cover of Γ^*, and hence S is transitive on $V\Gamma^*$ and the 2-arcs of Γ^*. Let $B := \alpha^C$, the C-orbit containing α. Let $S^* := (C \times S)/C \cong S$ and $G^* := (C.G)/C \cong G$. Set $\Gamma^* = \Gamma(S^*, S_B^*, S_B^* g^* S_B^*)$, the quotient graph of Γ modulo C-orbits, where $S_B^* \cong T{:}3$ and $g^* = Cg$ with $g \in \Delta(S, Q_8)$. Now S^* and G^* are subgroups of X^*. Further, by Lemma 2.1, S_B^* is a maximal subgroup of S^*, and so $\mathrm{C}_{X^*}(S^*) = 1$ by [9, Proposition 2.4]. If $soc(X^*) \neq S^*$, by [10, Theorem 1.1], either X^* has an almost simple subgroup Y^* with socle R^* such that $R^* \neq S^*$ and G^* is a maximal subgroup of Y^*, or X^* contains a subgroup $Y^* = Z_2^d{:}S^*$ such that $2^d = q^3$ and Z_2^d is semiregular on $V\Gamma^*$. In the latter case $|S_B^*|$ must be odd, which contradicts $S_B^* \cong Z_3^2{:}Q_8{:}3$. Now X^* has an almost simple subgroup Y^* with socle R^* satisfying the properties that: $R^* \neq S^*$ and G^* is a maximal subgroup of Y^*. By [21, 22], $soc(Y^*) = A_n$ and $Y^* = AG_1^*$ where G_1^* is an almost simple group with socle S^* and A is a subgroup of Y^* not containing A_n such that either $A_{n-k} \lhd A \leq S_{n-k} \times S_k$ with G^* acting k-homogeneously of degree n, or n is 6, 8 or 10. Clearly the latter case does not occur. In the former case, by [16], S has no k-homogeneous representation for $k \geq 3$ (see also [15, Chapter XII, Section 6]). Hence, we have $k \leq 2$. Also $n \geq m(S) = q^3 + 1$, where $m(S)$ denotes the minimal index of proper subgroups of S. Recall that the valency of Γ^* is 9 (since Γ is a cover of Γ^*). Thus every prime divisor of $|X_B^*|$ lies in $\{2, 3, 5, 7\}$. Moreover, since $X^* = X_B^* S^*$ and since the largest prime divisor of $|S^*|$ is less than q^2 (since $|S^*| = q^3(q^2-1)(q^3+1)/(3, q+1)$ and

$q^3+1=(q^2-q+1)(q+1)$), it follows that the largest prime divisor of $|X^*|$ is less than q^2. However, it is trivial to see that there is a prime p with $q^2 < p < q^3+1$, and hence $|X^*|$ is divisible by p, which is a contradiction. Thus $soc(X^*) = S \cong S^*$. □

Theorem 1.2 follows immediately from Propositions 3.1, 3.2, 2.3 and Theorem 1.1.

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XIN GUI FANG AND GEORGE HAVAS
Department of Mathematics,
Peking University,
Beijing 100871,
P. R. China

JIE WANG
Centre for Discrete Mathematics and Computing,
Department of Computer Science and Electrical Engineering,
The University of Queensland,
Queensland 4072,
Australia