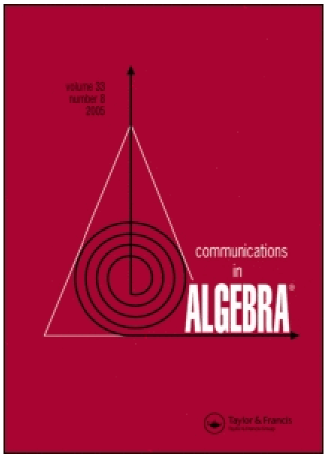

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Groups with exponent six

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GROUPS WITH EXPONENT SIX

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Abstract

Burnside asked questions about periodic groups in his influential paper of 1902. The study of groups with exponent six is a special case of the study of the Burnside questions on which there has been significant progress. It has contributed a number of worthwhile aspects to the theory of groups and in particular to computation related to groups. Finitely generated groups with exponent six are finite. We investigate the nature of relations required to provide proofs of finiteness for some groups with exponent six. We give upper and lower bounds for the number of sixth powers needed to define the largest 2-generator group with exponent six. We solve related questions about other groups with exponent six using substantial computations which we explain.

1 Introduction

Burnside [2] considered the freest group on d generators with exponent n (denoted $B(d,n)$) and asked the question: is $B(d,n)$ finite and, if so, what is its order? Starting from Burnside, many people have investigated aspects of these and related questions in detail. Presently it is well-known that $B(d,n)$ is finite if n is 1, 2, 3, 4 or 6 (or if d is 1), and no others are known to be finite (see Vaughan-Lee [23]). The group $B(d,n)$ is known to be infinite for $d \geq 2$ and $n \geq 8000$ (see Ivanov [13] and Lysënok [15]).

As well as asking for the order of finite Burnside groups one can ask about presentations for them and, in particular, presentations with the appropriate power relations. Such questions provide some of the motivation for this paper. For example, it is known that $B(2,4)$ can be defined by 9 fourth powers and needs at least 9 relators to define it, see Macdonald [16]. Here we study presentations for groups with exponent six. This is partly in the hope that it might give insights to help with the study of $B(2,5)$.

The Burnside two-prime theorem [3] yields that finite groups with exponent six are soluble. P. Hall and Higman [7] showed that all finite groups with exponent six have 2-length and 3-length at most 1. Consequently these groups are metanilpotent and the restricted Burnside problem (which asks if there is a largest finite group with given exponent) has a positive solution for groups with exponent six. Hall and Higman also determined the order of the largest finite quotient of $B(d,6)$. M. Hall [4] showed that the groups $B(d,6)$ are finite. For an outline see Hall's book [6]. More recent proofs of finiteness have been given by Newman [18] and Lysënok [14] (see Vaughan-Lee [23]).

We begin by considering general and polycyclic presentations. We give a polycyclic presentation for $B(2,6)$ (Section 2). We also discuss other presentations for $B(2,6)$. In particular we show that it has a presentation on 2 generators with 81 relations.

A motivating question is to get estimates for the number and length of sixth power relations which suffice to define $B(2,6)$. We show that M. Hall's finiteness proof yields that less than 2^{124} sixth powers can define $B(2,6)$ (Theorem 2). On the other hand the best lower bound we have proved is that at least 22 sixth powers are needed (Theorem 1). We expect that 22 is closer to the truth than 2^{124}. We observe that current proof methods cannot yield fewer than 2^{95} sixth powers in a defining set of relators.

This paper is a first step in the direction of finding 'short' sets of sixth powers using the best currently available computer programs. Further direct progress with $B(2,6)$ is a challenging problem. It seems unlikely that the computer-based methods now available — coset enumeration and Knuth-Bendix string rewriting — will succeed either in proving $B(2,6)$ itself is finite or in finding 'small' defining sets in the foreseeable future because the order of $B(2,6)$ is $2^{28}3^{25}$. It also seems unlikely that much more elaborate computational procedures based on coset enumeration will succeed in the near future. But perhaps there is hope for a judicious mixture of hand and machine such as that used by Vaughan-Lee [24] to handle a presentation for a group of order 5^{145}.

Because of this we consider some quotients of $B(2,6)$ for which coset enumeration and rewriting methods enable progress to be made. More precisely we consider group presentations $\{ X \mid \mathcal{R} \}$ with $X, \mathcal{R}$ finite to which the exponent six condition is added; we write $\{ X \mid \mathcal{R}, \exp 6 \}$. Such groups are finite and so a finite set of sixth power relators suffices to enforce the exponent six condition. Thus we study presentations $\{ X \mid \mathcal{R}, \mathcal{S} \}$ where $\mathcal{S}$ is a finite set of sixth powers and seek 'small', even irredundant or minimal, sets $\mathcal{S}$ for which the groups $\langle X \mid \mathcal{R}, \mathcal{S} \rangle$ and $\langle X \mid \mathcal{R}, \exp 6 \rangle$ defined by the presentations are isomorphic.

Let $C(r,s)$ denote the largest 2-generator group with exponent six generated by elements of orders r and s. To understand $B(2,6)$ better we look at presentations for the groups $C(2,2)$, $C(2,3)$, $C(3,3)$, $C(2,6)$ and $C(3,6)$. Let $\{a,b\}$ be a generating set for $B(2,6)$. The subgroup $H = \langle a^{6/r}, b^{6/s} \rangle$ of $B(2,6)$ is clearly a quotient group of $C(r,s)$. It turns out that H and $C(r,s)$ are isomorphic. The order of H and the index of the normal closure in $B(2,6)$ of $\langle a^r, b^s \rangle$ are easily computed using a polycyclic presentation for $B(2,6)$. In each case these numbers are the same.

The major computational tools we use are outlined in [10]. We exhibit current capabilities of various computational methods and illustrate the use of a powerful new tool, a soluble quotient program, see Niemeyer [20]. Coset enumeration can now routinely handle millions of cosets, see Havas [8]. For rewriting (see [22, Chapter 2]) we use the Rutgers Knuth-Bendix Package (RKBP) written by Sims. Such programs are available as standalone packages [10] and in systems such as GAP [21], MAGMA [1] and **quotpic** [11]. The various minor calculations which arise in this paper are done using one of these systems.

The presentation $\{ a,b \mid a^2, b^2, (ab)^6 \}$ is a minimal presentation for the group $C(2,2)$, the dihedral group of order 12. A minimal presentation for $C(2,3)$, a group of order 216, is $\{ a,b \mid a^2, b^3, (ab)^6, [a,b]^6 \}$. Therefore the first challenging case to consider is the group $C(3,3)$ (Section 4). It is a group of order $2^{10}3^3$. The general methods used in Section 2 show that this group has a presentation with

2 generators and 30 relations. The methods of Section 3 show that $C(3,3)$ can be presented as $\{\ a,b \mid a^3,b^3,\mathcal{S}\ \}$ with sets $\mathcal{S}$ consisting of at most 2^{55} sixth powers. On the other hand $C(3,3)$ certainly needs 5 relations to define it in view of the rank of its multiplier. We show that a set $\mathcal{S}$ which suffices to define $C(3,3)$ must contain at least 5 sixth powers (Theorem 9). Both coset enumeration and rewriting show that there is a set $\mathcal{S}$ of 11 sixth powers which suffices. With coset enumeration this has been reduced to 8 sixth powers, but we have been unable to close the gap. Rewriting reveals short relators such as $([b,a][b^{-1},a])^2$ which need longer sixth powers to establish. Such relators play an important simplifying role in later calculations (see Section 6) — we call them *subtle.*

Having discussed $C(3,3)$ in detail, we proceed, more briefly, to the groups $C(2,6)$ and $C(3,6)$. Using rewriting, we have shown that $C(2,6)$ has a presentation $\{\ a,b \mid a^2,\mathcal{S}\ \}$ with a set $\mathcal{S}$ of 15 sixth powers (Section 5). Also using rewriting, we have found a 'short' presentation for $C(3,6)$ (Section 6). However we do not have a concise set of sixth powers from which we can deduce this presentation.

2 Presentations for $B(2,6)$

Polycyclic presentations for $B(2,6)$ based on a composition series have 53 generators and 1431 relations. In the Appendix we give a presentation for $B(2,6)$ of this kind which shows much of the group's structure. It is a presentation of total length (the sum of the lengths of the words on the right hand sides) 6001. A verification that this presentation is a presentation for $B(2,6)$ is given towards the end of this section. Our presentation is available in both GAP and MAGMA formats. Stephen Glasby obtained a polycyclic presentation in 1990 and his presentation is available in MAGMA. It has total length 6563.

Because the Sylow 2-subgroup is elementary abelian of order 2^{28}, every polycyclic series has at least 28 factors. There are polycyclic series with 28 factors. For example, suppose $\{a,b\}$ is a generating set for $G := B(2,6)$ and G' is its commutator subgroup; begin a polycyclic series $G > G_1 > G_2 = G'$ with G/G_1 generated by aG_1 and G_1/G_2 generated by bG_2. Since G' is an extension of the nilpotent residual R of G by a cyclic group of order 3 and has non-trivial fixed-points in the Sylow 2-subgroup of R, there is a normal subgroup G_3 in G' with G'/G_3 cyclic of order 6. Since G_3 is nilpotent of order $2^{25}3^{22}$, it has a polycyclic series of length 25 with 22 factors of order 6 and 3 factors of order 2. A polycyclic presentation for $B(2,6)$ based on such a series has 28 generators and 406 relations.

Of these the 28 powers and the 53 conjugations by a and b are enough to define the group. We can express the generators of the factors G_i/G_{i+1} for $i \in \{2, \ldots, 27\}$ in terms of a, b. Then Tietze transformations can be used to give a presentation for $B(2,6)$ on 2 generators with 81 relations.

Theorem 1 *Every presentation for $B(2,6)$ with 2 generators has at least 20 relations and every presentation with only sixth power relations has at least 22 relations.*

PROOF. Let F be a free group of rank 2. Let Y be a set of generators for F^6 as a normal subgroup. Let $S = F^2$. Then S is free of rank 5 (Schreier's Theorem). Let $L = S^3$. Then S/L is $B(5,3)$ and has order 3^{25}. Let $N = [L,S]L^3$. Thus S/N is the 3-covering group of S/L. It is known that L/N is elementary abelian of order 3^{80} (see the Remark following Lemma 5). Since L/N is centralised by S, it can be viewed as a module for F/S over the field $GF(3)$. The set $\{yN \mid y \in Y\}$ is a normal generating set for L/N. So it is a module generating set. The free F/S-module has dimension 4 over K. Hence a module generating set for L/N has size at least 20. The first result follows.

Now let Y consist of sixth powers. Since $F^6[S,F]S^2 = S$, we get that Y contains (equivalents of) $(au)^6, (bv)^6, (abw)^6$ for some $u, v, w \in S$. Modulo N the centralisers of each of these three powers have index at most 2, therefore the submodules they generate have dimension at most 2 and so at least 19 more relations are needed. □

The power conjugate presentation $\{ \ \mathcal{A} \mid \mathcal{R} \ \}$ given in the Appendix is a presentation for $B(2,6)$. The generating set $\mathcal{A}$ comprises 53 elements $a_1, \ldots, a_{53}$. The presentation is consistent and was obtained using the ANU Soluble Quotient Program (ANU SQ) [20] and GAP. A verification that the presentation in the Appendix is a presentation for $B(2,6)$ consists of the following steps.

Firstly we show that the power conjugate presentation is consistent, hence the order of the group it defines is the product of the primes involved in the power relations and is $2^{28}3^{25}$, which is also the order of $B(2,6)$. Secondly we show that the presentation defines a group that can be generated by two elements. Finally we show that it defines a group with exponent six.

It is easy to check that the presentation is consistent. This can be done routinely in GAP or MAGMA. Therefore, the order of the group defined by the presentation is $2^{28}3^{25}$. The subgroup generated by

$$\{a_2a_5a_6a_7a_9a_{10}a_{13}a_{14}a_{15}a_{18}a_{29}a_{30}a_{32}a_{33}^2a_{35}^2a_{37}a_{39}a_{40}^2a_{42}a_{43}^2a_{45}^2a_{47}a_{48}a_{49}a_{50}^2a_{52}a_{53},$$
$$a_1a_4a_{11}a_{14}a_{16}a_{17}a_{19}a_{20}a_{22}a_{23}a_{24}a_{27}a_{29}a_{30}a_{34}a_{35}a_{39}a_{40}^2a_{41}a_{42}a_{44}^2a_{45}a_{46}^2a_{50}^2a_{51}^2a_{52}a_{53}^2\}$$

has order $2^{28}3^{25}$ and hence is the group itself. Thus the group is a 2-generator group.

The fact that the group described by the presentation has exponent six follows from the way it was constructed, which we now outline.

Let F be the free group of rank 2 on $\{a, b\}$. Then F has a normal subgroup F^3 which is the group generated by all cubes in F. This group is free of rank 28 and hence F^3/M, where $M = (F^3)^2$, is an elementary abelian 2-group of order 2^{28}. Let G_1 denote F/M. Then G_1 has order $2^{28}3^3$ and is a group with exponent six. Let G_2 denote F/L, where $L = (F^2)^3$. Then G_2 has order $2^2 3^{25}$ and exponent six. The group $B(2,6)$ is isomorphic to $F/(M \cap L)$ (see Huppert and Blackburn [12, p. 417]). Further, $F/(M \cap L)$ embeds into the direct product $F/M \times F/L$ via the natural embedding $\varepsilon : x(M \cap L) \mapsto (xM, xL)$. Clearly $F/M \times F/L$ is a group with exponent six, so the 2-generator subgroup of it generated by (aM, aL) and (bM, bL) must be isomorphic to $B(2,6)$.

Using ANU SQ we computed consistent power conjugate presentations for the groups G_1 and G_2 and epimorphisms $\kappa : F \to G_1$ and $\lambda : F \to G_2$. Then $B(2,6)$ is isomorphic to the subgroup of the direct product $G_1 \times G_2$ generated by (a^κ, a^λ) and (b^κ, b^λ). Hence by construction this subgroup is a group with exponent six, has two generators and the same order as $B(2,6)$.

The given power conjugate presentation for $B(2,6)$ exhibits some additional properties of the group. Since F/M has a normal elementary abelian subgroup F^3/M of order 2^{28}, this subgroup can be viewed as a module for F/F^3 over $GF(2)$. As such it decomposes into a direct sum of irreducible modules, namely two trivial modules, four 2-dimensional modules and three 6-dimensional (faithful) modules. Hence we can choose a power conjugate presentation for $B(2,6)$ which exhibits this module decomposition. Modulo M, generators a_4 and a_5 each generate a trivial F/F^3-module; the four 2-dimensional modules are generated by the sets $\{a_6, a_7\}$, $\{a_8, a_9\}$, $\{a_{10}, a_{11}\}$ and $\{a_{12}, a_{13}\}$; and the three 6-dimensional modules are generated by the sets $\{a_{14}, a_{15}, a_{16}, a_{17}, a_{18}, a_{19}\}$, $\{a_{20}, a_{21}, a_{22}, a_{23}, a_{24}, a_{25}\}$ and $\{a_{26}, a_{27}, a_{28}, a_{29}, a_{30}, a_{31}\}$.

Recall the *lower exponent-p central series* of a group G is the descending sequence of subgroups

$$G = \mathcal{P}_0^p(G) \geq \mathcal{P}_1^p(G) \geq \cdots \qquad \text{with } \mathcal{P}_i^p(G) = [\mathcal{P}_{i-1}^p(G), G](\mathcal{P}_{i-1}^p(G))^p \text{ for } i \geq 1.$$

(See [22, p. 445] where a different notation is used.) If c is the smallest integer such that $\mathcal{P}_c^p(G) = \langle 1 \rangle$ then G has *exponent-p class c*.

The quotient F/L has a normal 3-subgroup $K = (F/L)^2$ of exponent-3 class 3 and order 3^{25}. The factor group $K/\mathcal{P}_1^3(K)$ can be viewed as a module for F/F^2 over $GF(3)$. As such it decomposes into a direct sum of 5 submodules, two trivial modules and three 1-dimensional ones. The latter can be described by generators which lie in the kernel of the action of a_4, a_5 and $a_4 a_5$, respectively. Hence we can choose a power conjugate presentation which exhibits this module decomposition modulo $\mathcal{P}_1^3(K)$. The quotient $\mathcal{P}_0^3(K)/\mathcal{P}_1^3(K)$ has order 3^5 and $\mathcal{P}_1^3(K)/\mathcal{P}_2^3(K)$ has order 3^{10}. The generators of the two trivial modules are a_1 and a_2 and the generators of the three 1-dimensionals are a_{32}, a_{33} and a_{34} modulo $\mathcal{P}_1^3(K)$.

3 Defining sets of sixth powers

Because $B(d,6)$ is finite, the following well-known argument shows that it has a presentation with d generators and finitely many sixth powers. Let F be a free group of rank d. The subgroup F^6 generated by sixth powers has finite index and so is finitely generated (Schreier's Theorem). Each element of F^6 can be written as a product of sixth powers, so F^6 can be generated by finitely many sixth powers. This proof does not directly give an explicit set of sixth powers which generates F^6 (as a normal subgroup) nor even an easily accessible estimate for the size of a set of sixth powers which suffices to generate it.

We can use this result to produce an algorithm which will yield an explicit set of sixth powers. Let X be a free generating set for F:

- start making a list of finite sets of sixth powers of words in X;
- for each set $\mathcal{S}$ in the list begin a coset enumeration on the presentation $\{X \mid \mathcal{S}\}$;
- run the coset enumerations in the 'usual' diagonal fashion;
- when one of the enumerations completes with the correct order (which must eventually happen) stop the whole process and print the set which led to the successful enumeration.

This is, of course, hopelessly impractical. Further, there is no guarantee that such a process will produce minimal or even irredundant sets of sixth powers, because it may not be possible to decide whether a given presentation with sixth power relators defines a finite group or not.

Theorem 2 *The group $B(2,6)$ can be defined by a presentation with 2 generators and less than 2^{124} sixth-power relators.*

The result follows from a sequence of lemmas, most of which appear in some form in the paper of M. Hall [5]. Let F be a free group of rank 2 freely generated by $\{a,b\}$. For brevity we often use the case inverse convention. Thus a^{-1} and b^{-1} may be denoted by A and B, respectively. The commutator $a^{-1}b^{-1}ab = ABab$ is denoted $[a,b]$. For convenience we allow mixtures of relators and relations in presentations.

Lemma 3 *Let R be the normal closure in F of $\{a^6, b^6, (ab)^6, (aB)^6\}$, then F^3/R can be generated by 28 elements of order 2.*

PROOF. A set of cubes generating F^3 can be found in the following way. Using the ANU SQ we obtain a consistent power conjugate presentation for $G = F/(F^3)^2$. We define subgroups of G generated by sets of cubes and their conjugates until we have defined a subgroup equal to $F^3/(F^3)^2$. We obtain: $\{$ a^3, b^3, b^{3a}, a^{3b}, $(ab)^3$, $(aB)^3$, $(ba)^3$, $(Ba)^3$, b^{3A}, a^{3ba}, b^{3ab}, a^{3B}, $(ba)^{3a}$, $(Ba)^{3a}$, $(ab)^{3b}$, $(Ba)^{3b}$, a^{3bA}, a^{3Ba}, b^{3Ab}, a^{3bab}, $(ab)^{3ba}$, $(Ba)^{3ba}$, $(ba)^{3ab}$, $(Ba)^{3ab}$, $(ab)^{3bA}$, $(Ba)^{3aba}$, $(ab)^{3bab}$, $(ab)^{3baba}$ $\}$. Coset enumeration in F over the subgroup generated by this set shows that the subgroup is F^3. □

The next lemma is an explicit form of Lemma 2 of Hall.

Lemma 4 *Let M be a group with exponent six generated by a finite number t of elements of order 2. Then M' can be generated by a set of $(t-2)2^{t-1}+1$ elements of the form $(xy)^2$ where x, y are elements of order 2 in M.*

PROOF. Let $\{m_1, \ldots, m_t\}$ be a (minimal) generating set for M consisting of elements of order 2. A transversal for M' in M is the set of products $m_1^{\delta_1} \ldots m_t^{\delta_t}$ where δ_i is 0 or 1. It is straight-forward to check that the elements $(m_i m_j)^{2vu}$ where $i < j$, $v = m_{j-1}^{\delta_{j-1}} \ldots m_{i+1}^{\delta_{i+1}}$ and $u = m_{i-1}^{\delta_{i-1}} \ldots m_1^{\delta_1}$ generate M'. □

The next lemma is essentially Theorem 2.2 of Hall.

Lemma 5 *Let $\mathcal{S} = \{$ a^3, b^3, c^3, d^3, $(ab)^3$, $(aB)^3$, $(ac)^3$, $(aC)^3$, $(ad)^3$, $(aD)^3$, $(bc)^3$, $(bC)^3$, $(bd)^3$, $(bD)^3$, $(cd)^3$, $(cD)^3$, $(abc)^3$, $(Abc)^3$, $(aBc)^3$, $(abC)^3$, $(abd)^3$, $(Abd)^3$, $(aBd)^3$, $(abD)^3$, $(acd)^3$, $(Acd)^3$, $(aCd)^3$, $(acD)^3$, $(bcd)^3$, $(Bcd)^3$, $(bCd)^3$, $(bcD)^3$ $\}$.*

(a) *The group presented by $\{ a,b,c,d \mid \mathcal{S} \}$ has order 3^{17}.*

(b) *The group presented by $\{ a,b,c,d \mid \mathcal{S}, (ab[c,d])^6, (ac[d,b])^6, (ad[b,c])^6 \}$ has exponent three and order 3^{14}.*

PROOF. These results are now easily derived by computation. For (a), coset enumeration shows that $\langle a,b,c\rangle$ has index 3^{10} and that $|\langle a,b,c\rangle|$ divides 3^7. The ANU p-Quotient Program [19] shows that the group has a quotient of order 3^{17}. For (b), coset enumeration shows that $\langle a,b,c\rangle$ has index 3^7 and that $|\langle a,b,c\rangle|$ divides 3^7. The ANU p-Quotient Program shows that the group has a quotient of order 3^{14} which has exponent three. □

Remark. The group $B(4,3)$ needs at least 35 relators to define it (this can be read off the presentation for its 3-covering group). By replacing the sixth powers in Lemma 5(b) by the corresponding cubes we obtain a minimal presentation for $B(4,3)$. Since $B(d,3)$ has class 3, it follows that $B(d,3)$ needs $d+2\binom{d}{2}+4\binom{d}{3}+3\binom{d}{4}$ relators to define it and no fewer.

Lemma 6 *In the group presented by*

$$\begin{aligned}\{\ & x, a_1, \ldots, a_n \mid x^2, (xa_i)^2, a_i^3, (a_ia_j)^6, (a_iA_j)^6, (xa_ia_j)^6, (x(a_ia_j)^3)^6,\\ & (a_ia_jA_k)^6, (a_iA_ja_k)^6, (A_ia_ja_k)^6,\\ & (a_ia_ja_ia_k)^6, (a_ia_ja_iA_k)^6, (a_iA_ja_ia_k)^6, (a_iA_ja_iA_k)^6,\\ & (xa_ia_ja_ia_k)^6, (xa_iA_ja_ia_k)^6, (x(a_ia_ja_ia_k)^3)^6, (x(a_iA_ja_ia_k)^3)^6,\\ & (a_ia_j[a_k,a_l])^6, (a_ia_k[a_l,a_j])^6, (a_ia_l[a_j,a_k])^6 \quad \text{for } 1 \le i < j < k < l \le n\ \}\end{aligned}$$

the subgroup $\langle a_1, \ldots, a_n\rangle$ *has exponent three.*

PROOF. The case $n = 2$ is easily done by coset enumeration. This enables us to replace the relations $(a_ia_j)^6$, $(a_iA_j)^6$, $(xa_ia_j)^6$ and $(x(a_ia_j)^3)^6$ by $(a_ia_j)^3$ and $(a_iA_j)^3$, and do the case $n = 3$ by coset enumeration. For $n = 4$ use Lemma 5(b) above. This shows that for $n > 4$ every 4-element subset of $\{a_1, \ldots, a_n\}$ generates a subgroup which is nilpotent of class 3 and the result follows. □

Remarks. This is Lemma 7 of Hall. The cases $n = 2$ and $n = 3$ are handled in [18]. In fact the subgroup is $B(n,3)$ and so the group has order $2 \times 3^{w(n)}$, where $w(n) = \binom{n}{1} + \binom{n}{2} + \binom{n}{3}$. The number of sixth powers is $4\binom{n}{2} + 11\binom{n}{3} + 3\binom{n}{4}$. The above presentation is close to minimal. The use of the relators $(a_ia_j)^3$ is an example of the use of subtle relators, which plays a critical role in Section 6.

Lemma 7 *Every group M with exponent six generated by a finite set of elements of order 2 is finite.*

PROOF. Let M have a generating set with t elements of order 2. Let $\mathcal{H}$ be a generating set for M' as provided by Lemma 4 and let $k = (t-2)2^{t-1}+1$. The critical case is $t = 6$ (cf. Lemma 3 of Hall). Lemma 3 of Hall and its antecedants can be proved with a set of less than 2000000 sixth powers. Most of these sixth powers come from an application of Lemma 6 above with $n = 54$. Thus there is a set of at most $2000000\binom{k}{3}$ sixth powers which shows that every 3-element subset of $\mathcal{H}$ generates a group with exponent three. It follows from Lemma 5(b) that there is a set of $2000000\binom{k}{3} + 3\binom{k}{4}$ sixth powers which yields that M' has exponent three. □

Remarks. When M is the freest group with exponent six generated by t elements of order 2, the commutator subgroup M' is free with exponent three on k generators. Hence, using the Remark after Lemma 5, M' needs at least $3\binom{k}{4}$ relators to define it. Therefore M needs at least $3\binom{k}{4}/2^t$ relators to define it. Thus this method of proof cannot be used to improve substantially the form of the upper bound for the number of sixth powers. In particular, for $t = 28$ one needs a set of at least 2^{95} sixth powers to show M' has exponent three. So at least 2^{95} sixth powers are needed to get a preimage G of $B(2,6)$ with $(G^3)^2$ having exponent three. Thus proofs of this kind give presentations for $B(2,6)$ with at least 2^{95} sixth powers.

Simply to prove the finiteness of $B(2,6)$ one can use Lemma 5(a) in the proof of Lemma 7. But even this part of the proof uses at least 2^{70} sixth powers (and yields a bigger group which is even more unattractive from a computational view).

4 The group $C(3,3)$

As pointed out in the introduction, the first interesting quotient of $B(2,6)$ obtained by applying order conditions on the generators is the factor group $C(3,3)$ given by the presentation $\{ a, b \mid a^3, b^3, \exp 6 \}$. Computation in $B(2,6)$ shows that $C(3,3)$ has order $2^{10}3^3 = 27648$. It is metanilpotent and has $B(2,3)$ as a quotient. The kernel of this quotient is elementary abelian and the direct sum of three irreducible $B(2,3)$-modules with dimensions 2,2,6.

The rank of the multiplier of $C(3,3)$ is at least three and hence it needs at least five relations to define it. Indeed minimal presentations exist, for example the following which is based on a presentation found by Peter Kenne:

$$\{ a, b \mid a^3, b^3, ([a,b][A,b])^2, abaBaBabABABAbAbABab, (abaBaBABaB)^2 \}.$$

That this is a presentation for $C(3,3)$ is easily verified using coset enumeration or rewriting. We show (Theorem 9) that every presentation with the two natural relators a^3 and b^3 and otherwise sixth powers needs at least seven relators.

A presentation for $C(3,3)$ obtained along the lines of the previous section would use Lemma 4 with $t = 10$ and have at most 2^{55} sixth powers and need at least 2^{35} sixth powers. Next we find a lower bound for the number of sixth powers necessary to define $C(3,3)$.

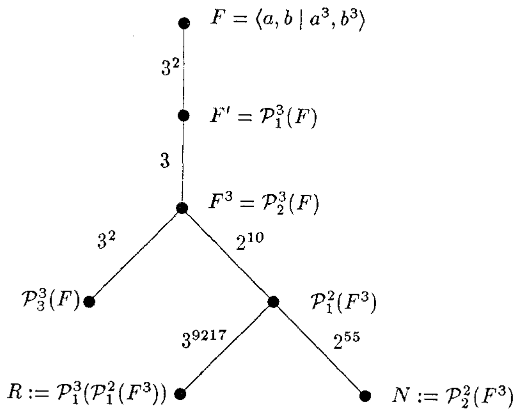

Figure 1: Quotients of F

Lemma 8 *A presentation for $C(3,3)$ needs at least 2 non-sixth-power relations.*

PROOF. The largest abelian quotient of $B(2,6)$ is $C_6 \times C_6$. The largest abelian quotient of $C(3,3)$ is $C_3 \times C_3$. □

Remark. This justifies our choice of the starting relators in our presentation study. We certainly need two non-sixth powers. Choices other than a^3, b^3 for two non-sixth-power relators are possible and lead to interesting problems. They are not discussed further in this paper.

Theorem 9 *Every presentation on the generating set $\{a, b\}$ with relators a^3 and b^3 and four sixth powers defines a group whose largest soluble quotient has order more than $2^{10}3^3$.*

PROOF. Clearly the group defined by such a presentation has $C(3,3)$ as a quotient. Consider the groups $F = \langle a, b \mid a^3, b^3 \rangle$ and $G = \langle a, b \mid a^3, b^3, w_1^6, w_2^6, w_3^6, w_4^6 \rangle$, and see Figure 1.

Let K denote the kernel of the epimorphism from F onto G and let N denote $\mathcal{P}_2^2(F^3)$. Then $K \leq \mathcal{P}_1^2(F^3)$. If $\mathcal{P}_3^3(F)K < F^3$, then 3^4 divides the order of G and the result holds. (In this case G has too large a nilpotent quotient.)

Now consider $\mathcal{P}_3^3(F)K = F^3$. Since $[K, F] \leq \mathcal{P}_3^3(F)$, it follows that $\mathcal{P}_3^3(F)K = \langle \mathcal{P}_3^3(F), w_1^6, w_2^6, w_3^6, w_4^6 \rangle$.

An element of F whose cube does not lie in $\mathcal{P}_3^3(F)$, or the inverse of the element, has the form $ab^{\pm 1}w$, where $w \in F'$. The elements $(abu)^3$ and $(aBv)^3$ are independent modulo $\mathcal{P}_3^3(F)$ for all $u, v \in F'$. Hence, without loss of generality, we can assume that $w_1 = abu$ and $w_2 = aBv$ for some $u, v \in F'$.

If $NK < \mathcal{P}_1^2(F^3)$, then 2^{11} divides $|G|$, and the result holds. (In this case G has too large a metanilpotent quotient.)

It remains to consider $NK = \mathcal{P}_1^2(F^3)$. Note that the normal closure in F/N of an element of the form $(wN)^6$ with $wN \notin (F/N)^3$ has order at most 2^9 (because its centraliser contains F^3/N and wN, and so has index at most 9 in F/N). Since $\mathcal{P}_1^2(F^3)/N$ is elementary abelian of order 2^{55}, it follows that $w_3, w_4 \in F^3$, and $w_3^6, w_4^6 \in \mathcal{P}_1^3(\mathcal{P}_1^2(F^3))$. Let $R := \mathcal{P}_1^3(\mathcal{P}_1^2(F^3))$. Note for $w \notin F'$ the normal closure of $(wR)^6$ in F/R has order at most $3^{9\times 2^9}$ (because its centraliser contains $\mathcal{P}_1^2(F^3)/R$ and wR). As the order of $\mathcal{P}_1^2(F^3)/\mathcal{P}_1^3(\mathcal{P}_1^2(F^3))$ is 3^{9217} and $9217 = 9 \times 2^{10} + 1$, it follows that G has a soluble quotient of order at least $3^{4 2^{10}}$. □

Methods similar to those in the proof allow us to use a polycyclic presentation for $Q := F/N$, whose order is $3^3 2^{65}$, to find short presentations $\{a, b \mid a^3, b^3, w_1^6, \ldots, w_n^6\}$ which define groups whose largest soluble quotient is $C(3,3)$.

By Theorem 9 we need at least 5 relations w_i^6 to obtain such a presentation. We describe below how to construct such presentations in practice. Note that we need to ensure at least that two of the powers give the group a largest nilpotent quotient of order 27 and the images of the powers in Q span Q^6. An example is

$$\Pi_0 = \{\ a, b \mid a^3, b^3, (ab)^6, (aB)^6, (abAb)^6, (abAB)^6, (ababABAbABaB)^6\ \}.$$

Here the sum of the lengths of the bases of the sixth powers is 24. No presentation in which the length of bases of the sixth powers in $\mathcal{S}$ is less than 24 has $C(3,3)$ as its largest soluble quotient.

We outline our construction. We found short words in the generators of Q whose sixth powers generate Q^6 in the following way. Let g be an element of Q. Then $g^6 = (g^3)^2$. Hence g^6 is a square of an element of Q^3. All elements of Q^3 have order either 2 or 4 and hence the set of sixth powers of elements of Q is the set of squares of the elements in Q^3. Further, for $x = g^3$ the conjugacy class $(x^2)^Q = (x^Q)^2$. It therefore suffices to consider representatives of the conjugacy classes of $Q^3/\mathcal{P}_1^2(Q^3)$ and compute their squares to obtain representatives of the conjugacy classes of sixth powers in $\mathcal{P}_1^2(Q^3)/\mathcal{P}_2^2(Q^3)$. There are 47 conjugacy classes of nontrivial elements in $Q^3/\mathcal{P}_1^2(Q^3)$ under the action of Q. For each representative $x\mathcal{P}_1^2(Q^3)$ we computed the size of the normal closure of x^2 in Q. In order to find enough sixth-power relations to obtain a presentation which is a candidate for having largest finite quotient $C(3,3)$, we need to find enough elements of the form x^2 for $x \in Q^3$, $x \notin \mathcal{P}_1^2(Q^3)$ such that the normal closure of the subgroup they generate is $\mathcal{P}_1^2(Q^3)$.

Testing whether a presentation has a largest soluble quotient of order $2^{10}3^3$ is done in the following way. Let P be the group defined by the presentation. First check that the largest nilpotent quotient of P has order 27, then check that the largest metanilpotent quotient of P has order $2^{10}3^3$. Finally, if P has passed these tests, then P^3 has 1023 subgroups of index 2, which fall into 47 conjugacy classes under the action of P. Using the ANU p-Quotient Program and **quotpic** we were able to compute a presentation for P^3. Using a **quotpic** script written by Derek Holt we then computed the abelian quotient invariants for the representatives of maximal subgroups of P^3. If each representative has largest abelian quotient elementary with order 2^9, then $C(3,3)$ is the largest soluble quotient of P. Significant speed-up for this last step was obtained by stopping a computation as soon as an abelian invariant other than 2 appeared.

To find upper bounds we start by taking short sixth powers and use coset enumeration and rewriting. We consider words in the free product F of two cyclic groups of order 3 defined by $\{a, b \mid a^3, b^3\}$. We construct words as alternating products of $a^{\pm 1}$ and $b^{\pm 1}$. It suffices to take one representative from each class of conjugates and inverses, normalised to start with a.

Let $\mathcal{S}_1 = \{\ (ab)^6,\ (aB)^6,\ (abaB)^6,\ (abAb)^6,\ (abAB)^6,\ (ababAB)^6,\ (abaBAb)^6,$ $(abaBAB)^6,\ (abAbaB)^6,\ (abAbAB)^6,\ (abABaB)^6\ \}$. This includes one representative of all words of length up to six which are different as potential relators for $C(3,3)$. We proved $\Pi_1 = \{\ a, b \mid a^3, b^3, \mathcal{S}_1\ \}$ is a presentation for $C(3,3)$ by coset enumeration and by rewriting.

Applying RKBP to Π_1 we obtain a confluent presentation from which we can compute the group order. In addition, we can extract short relations from the confluent presentation. In this context, we say a relator is *subtle* if it is shorter than the longest sixth power required for a proof that it holds, starting with a presentation including only the initial relators ($\{a^3, b^3\}$ in this case) and sixth powers. In $C(3,3)$, the relators $([b,a][B,a])^2$ and $([a,b][A,b])^2$ are subtle. In fact, the following is a much more concise presentation of this group:

$$\Pi_2 = \{\ a, b \mid a^3, b^3, (ab)^6, (aB)^6, ([b,a][B,a])^2, ([a,b][A,b])^2\ \}.$$

Furthermore, coset enumeration can find the 27648 cosets over the trivial subgroup without defining any redundant cosets for this presentation.

Coset enumeration takes as input a group given by a finite presentation and a finitely generated subgroup of it, and, if the subgroup has finite index, gives as output the index. For our purposes we need to be able to find a finitely generated subgroup where the coset enumeration completes and where we can prove the subgroup is finite. The simplest case is to take the trivial subgroup. The next best case is to take cyclic subgroups. If one can be found with finite index, then the Reidemeister-Schreier algorithm [17] can be used to find a presentation for it. Its order can then be calculated using integer matrix diagonalization [9] since the subgroup, being cyclic, is abelian.

Coset enumerations of Π_1 over cyclic subgroups have proved difficult. Since we attempt to prove finiteness for many related finitely presented groups by coset enumeration, we use a more sophisticated technique.

We are examining groups $G(\mathcal{S})$ given by presentations $\{a, b \mid a^3, b^3, \mathcal{S}\}$ where $\mathcal{S}$ is a finite set of sixth powers. Since we can compute soluble quotients of such groups, we can enumerate over soluble subgroups. Note that if $\mathcal{S}$ contains $(ab)^6$ and $(abaB)^6$, then the subgroup T generated by $\{t = A, u = (ab)^3\}$ satisfies the relators $t^3, u^2, (tu)^6$ and so is a triangle group which is soluble. Thus, if we can prove that T has index a $\{2,3\}$-number, then $G(\mathcal{S})$ is soluble. In Π_1 the subgroup T has index $1152 = 2^7 3^2$ and Π_1 has $C(3,3)$ as its largest soluble quotient, so Π_1 presents $C(3,3)$. With the coset enumerator in MAGMA, performance statistics for the enumeration Π_1 over T are: index 1152 with a maximum of 3245801 and total of 3417675 cosets. (This enumeration and all difficult enumerations with reported performance were completed using the procedure defined by the MAGMA parameters `Strategy := <4300,5>` and `SubgroupRelations := 2`.) The index 4608 enumeration Π_1 over $\langle aB \rangle$ completes with a maximum of 21419244 and total of 21428305 cosets.

Now that we have a reasonably short presentation for $C(3,3)$, we attempt to find shorter ones by systematically deleting sixth powers and trying permutations and formal inversion of relators. Before embarking upon what may be a very difficult and resource consuming coset enumeration, we check that the group defined has a largest soluble quotient with order $2^{10}3^3$.

Let $\mathcal{S}_2 = \{\ (ab)^6, (aB)^6, (abaB)^6, (abAb)^6, (abAB)^6, (abaBAb)^6, (aBaBAb)^6, (aBabAB)^6\ \}$. We find that the coset enumeration over T in $\{a, b \mid a^3, b^3, \mathcal{S}_2\}$ completes with index 1152 using a maximum of 4749142 and total of 4751935 cosets. This is the best enumeration that we have found for proving finiteness for a ten relator presentation for $C(3,3)$ which includes 8 sixth powers. A similar argument shows that T is finite in this context. Here the index 4608 enumeration over $\langle aB \rangle$ completes with a maximum of 85090780 and total of 85659443 cosets.

Three presentations obtained by excluding one of these 8 sixth powers give groups whose maximal soluble quotient is isomorphic to $C(3,3)$ (exclude one of $(aB)^6$, $(aBaBAb)^6$ and $(aBabAB)^6$). We have not been able to determine whether (or not) any of these presentations or Π_0 defines $C(3,3)$. Excluding any other single sixth power or any two sixth powers from $\mathcal{S}_2$ gives groups with larger soluble quotients.

5 The group $C(2,6)$

Now we study $C(2,6)$ given by the presentation $\{ a, b \mid a^2, \exp 6 \}$. This group has order $2^4 3^7 = 34992$. Using similar techniques to those in the previous section we construct sets of short sixth powers. Consider $\mathcal{S}_3 = \{ b^6, (ab)^6, (abb)^6, (abaB)^6, (abbb)^6, (ababb)^6, (abaBB)^6, (abbaBB)^6, (ababaB)^6, (ababbb)^6, (ababaBB)^6, (ababbaB)^6, (abaBabb)^6, (abbabbb)^6, (ababaBaB)^6 \}$. Using rewriting we can show that $\Pi_3 = \{a, b \mid a^2, \mathcal{S}_3\}$ is a presentation for $C(2,6)$.

This is not a trivial computation. The longest rule in the confluent rewriting system for $C(2,6)$ has a left side of length 22. RKBP appears to need to form overlaps of length about 32 to obtain the final confluent system from the above presentation.

The sixth power of at least one word of length greater than 7 is needed to define $C(2,6)$ this way. The presentation $\{a, b \mid a^2, w^6 \text{ for } |w| \leq 7\}$ defines a group with order $3|C(2,6)|$.

There are many subtle relators for $C(2,6)$, including the non-sixth powers (other than a^2) in the following short presentation for the group:

$$\begin{aligned}\Pi_4 = \{ a, b \mid & a^2, b^6, (ab)^6, (abb)^6, (ababbb)^3, (abaB)^6, (aBabaBabbabb)^2, \\ & (abaBaBBabbabbb)^2, bbaBabaBBababa = ababaBBabaBabb \}.\end{aligned}$$

Coset enumeration in Π_4 is easy. In Π_3 coset enumeration finds the 5832 cosets of $\langle b \rangle$ with a maximum of 22298313 and total of 22308499 cosets.

6 The group $C(3,6)$

The group $C(3,6)$ is much more difficult to work with computationally because it has order $2^{19}3^7 = 1146617856$. In the investigation of this group, we use some subtle relators we have obtained for $C(3,3)$ and $C(2,6)$ to deduce some relators for $B(2,6)$ which are difficult to derive directly from sixth powers.

Let $\{a, b\}$ be a generating set for $B(2,6)$. Then a^2 and b^2 satisfy all the relations for $C(3,3)$. From Π_2, we conclude that $([b^2, a^2][B^2, a^2])^2$ and $([a^2, b^2][A^2, b^2])^2$ are relators which hold in $B(2,6)$. Also, the elements a^3 and b in $B(2,6)$ satisfy the relations for $C(2,6)$. Substituting a^3 for a in the subtle relations for $C(2,6)$ in Π_4, we deduce that $(a^3ba^3bbb)^3$, $(a^3Ba^3ba^3Ba^3bba^3bb)^2$ and $(a^3ba^3Ba^3BBa^3bba^3bbb)^2$

are relators and $bba^3Ba^3ba^3BBa^3ba^3ba^3 = a^3ba^3ba^3BBa^3ba^3Ba^3bb$ is a relation in $B(2,6)$. Because $B(2,6)$ is a relatively free group, a relation satisfied by a and b is actually an identity for $B(2,6)$ and so consequences of it are also relations.

Further, let $\{a, b\}$ be a generating set for $C(3,6)$ with $a^3 = 1$. Then a and b^2 satisfy the relations for $C(3,3)$.

A reasonably short presentation for $C(3,6)$ in terms of sixth powers and the relation $a^3 = 1$ has not yet been determined. However, by using consequences of the subtle relations together with the sixth powers of all words of length up to 10 as input to RKBP it is possible to obtain a confluent presentation for the group. This involves about an elapsed week of computer time together with intelligent human interaction.

From the confluent presentation we obtain the following presentation for $C(3,6)$: $\Pi_5 = \{\, a, b \mid a^3,\ b^6,\ (ab)^6,\ (aB)^6,\ (baBBaBBa)^2,\ (bABBABBA)^2,\ (babAbbaBA)^2,$ $(babABabbA)^2,\ (babbabABA)^2,\ (babbAbABa)^2,\ (babbABabA)^2,\ (baBabAbbA)^2,$ $(abb)^6,\ (aBB)^6,\ babaBBAbAba = abAbABBabab,\ (abAb)^6,\ (abaB)^6,\ (ababbb)^6 \,\}$.

A computational verification that this is a presentation for $C(3,6)$ is not difficult. Coset enumeration of the 41472 cosets of $\langle a, b^2 \rangle$ in Π_5 is easy. RKBP has no trouble producing enough of the rules for the group defined by Π_5 to deduce that a and b^2 satisfy the short presentation for $C(3,3)$. Thus the order of the group defined is at most $41472 \times |C(3,3)| = |C(3,6)|$. Since the relations hold in $C(3,6)$, the presentation defines $C(3,6)$. The last relator $(ababbb)^6$ is definitely not a consequence of the earlier ones. Without it, the group defined has order $3|C(3,6)|$.

Appendix: A presentation for $B(2,6)$

The following power conjugate presentation $\{\mathcal{A} \mid \mathcal{R}\}$ is a presentation for $B(2,6)$. The generating set $\mathcal{A}$ comprises 53 elements $a_1, \ldots, a_{53}$. The generators satisfy the relations $a_i^3 = 1$ for $1 \le i \le 3$ or $32 \le i \le 53$ and $a_i^2 = 1$ for $4 \le i \le 31$. Most of the conjugate relations are of the form $a_j^{a_i} = a_j$ for $1 \le i < j \le 53$. Here are the other conjugate relations.

$$\begin{array}{ll}
a_2^{a_1} = a_2a_3 & a_4^{a_1} = a_4a_{42}^2a_{43}a_{47}a_{50}^2a_{52}^2a_{53}^2 \\
a_4^{a_2} = a_4a_{42}a_{51}^2a_{53} & a_4^{a_3} = a_4a_{48}a_{50}a_{52}^2 \\
a_5^{a_1} = a_5a_{41}a_{43}a_{47}a_{50}^2a_{51}a_{52}a_{53}^2 & a_5^{a_2} = a_5a_{41}^2a_{49}a_{51}a_{52}a_{53} \\
a_5^{a_3} = a_5a_{47}^2a_{49}^2a_{52}^2 & a_6^{a_1} = a_6a_7 \\
a_7^{a_1} = a_6 & a_8^{a_1} = a_8a_9 \\
a_8^{a_2} = a_9 & a_9^{a_1} = a_8 \\
a_9^{a_2} = a_8a_9 & a_{10}^{a_2} = a_{11}
\end{array}$$

$a_{11}^{a_2} = a_{10}a_{11}$
$a_{12}^{a_2} = a_{13}$
$a_{13}^{a_2} = a_{12}a_{13}$
$a_{14}^{a_2} = a_{14}a_{15}$
$a_{15}^{a_1} = a_{14}a_{15}a_{16}a_{17}a_{19}$
$a_{15}^{a_3} = a_{14}a_{17}a_{18}a_{19}$
$a_{16}^{a_2} = a_{14}a_{15}a_{16}$
$a_{17}^{a_1} = a_{14}a_{15}a_{17}a_{19}$
$a_{17}^{a_3} = a_{15}a_{16}a_{18}a_{19}$
$a_{18}^{a_2} = a_{17}a_{18}a_{19}$
$a_{19}^{a_1} = a_{14}a_{18}$
$a_{19}^{a_3} = a_{15}a_{16}a_{17}$
$a_{20}^{a_2} = a_{20}a_{22}a_{24}$
$a_{21}^{a_2} = a_{24}$
$a_{22}^{a_1} = a_{24}a_{25}$
$a_{22}^{a_3} = a_{21}a_{24}a_{25}$
$a_{23}^{a_2} = a_{20}a_{23}a_{24}a_{25}$
$a_{24}^{a_1} = a_{22}a_{24}a_{25}$
$a_{24}^{a_3} = a_{22}a_{24}a_{25}$
$a_{25}^{a_3} = a_{21}a_{25}$
$a_{26}^{a_2} = a_{26}a_{27}a_{28}a_{31}$
$a_{27}^{a_1} = a_{26}$
$a_{27}^{a_3} = a_{29}$
$a_{28}^{a_2} = a_{27}a_{30}a_{31}$
$a_{29}^{a_1} = a_{28}$
$a_{29}^{a_3} = a_{27}a_{29}$
$a_{30}^{a_2} = a_{27}a_{28}a_{30}a_{31}$
$a_{31}^{a_1} = a_{26}a_{30}$
$a_{31}^{a_3} = a_{26}a_{27}a_{29}a_{30}a_{31}$
$a_{32}^{a_2} = a_{32}a_{36}a_{53}^2$
$a_{32}^{a_4} = a_{32}^2a_{39}a_{46}a_{47}^2a_{48}a_{51}a_{53}$
$a_{33}^{a_1} = a_{33}a_{37}a_{53}$
$a_{33}^{a_3} = a_{33}a_{45}$
$a_{33}^{a_5} = a_{33}a_{36}^2a_{43}a_{46}a_{47}^2a_{49}^2a_{50}^2a_{53}$
$a_{34}^{a_1} = a_{34}a_{39}a_{53}^2$
$a_{34}^{a_3} = a_{34}a_{46}$
$a_{34}^{a_5} = a_{34}^2a_{45}^2a_{48}^2a_{50}^2$
$a_{34}^{a_{33}} = a_{34}a_{43}^2$

$a_{12}^{a_1} = a_{13}$
$a_{13}^{a_1} = a_{12}a_{13}$
$a_{14}^{a_1} = a_{15}a_{19}$
$a_{14}^{a_3} = a_{14}a_{15}a_{17}a_{19}$
$a_{15}^{a_2} = a_{14}$
$a_{16}^{a_1} = a_{14}a_{15}a_{16}$
$a_{16}^{a_3} = a_{14}a_{19}$
$a_{17}^{a_2} = a_{19}$
$a_{18}^{a_1} = a_{14}a_{15}a_{18}$
$a_{18}^{a_3} = a_{17}a_{19}$
$a_{19}^{a_2} = a_{18}a_{19}$
$a_{20}^{a_1} = a_{21}a_{23}$
$a_{20}^{a_3} = a_{20}a_{23}a_{24}$
$a_{21}^{a_3} = a_{25}$
$a_{22}^{a_2} = a_{23}a_{25}$
$a_{23}^{a_1} = a_{20}a_{21}a_{23}a_{25}$
$a_{23}^{a_3} = a_{20}a_{22}a_{24}a_{25}$
$a_{24}^{a_2} = a_{20}a_{21}a_{24}a_{25}$
$a_{25}^{a_2} = a_{22}a_{24}a_{25}$
$a_{26}^{a_1} = a_{26}a_{28}a_{29}$
$a_{26}^{a_3} = a_{28}$
$a_{27}^{a_2} = a_{26}a_{29}a_{30}$
$a_{28}^{a_1} = a_{26}a_{27}a_{29}$
$a_{28}^{a_3} = a_{26}a_{28}$
$a_{29}^{a_2} = a_{29}a_{31}$
$a_{30}^{a_1} = a_{26}a_{27}a_{29}a_{30}a_{31}$
$a_{30}^{a_3} = a_{27}a_{28}a_{31}$
$a_{31}^{a_2} = a_{26}a_{27}a_{29}a_{30}a_{31}$
$a_{32}^{a_1} = a_{32}a_{35}$
$a_{32}^{a_3} = a_{32}a_{44}$
$a_{32}^{a_5} = a_{32}^2a_{38}a_{45}^2a_{48}^2a_{49}a_{50}a_{52}^2a_{53}$
$a_{33}^{a_2} = a_{33}a_{38}a_{53}$
$a_{33}^{a_4} = a_{33}^2a_{46}a_{47}^2a_{53}$
$a_{33}^{a_{32}} = a_{33}a_{41}^2$
$a_{34}^{a_2} = a_{34}a_{40}a_{53}$
$a_{34}^{a_4} = a_{34}a_{35}^2a_{43}^2a_{45}^2a_{47}^2a_{50}^2a_{52}^2a_{53}^2$
$a_{34}^{a_{32}} = a_{34}a_{42}^2$
$a_{35}^{a_2} = a_{35}a_{44}$

$a_{35}^{a_4} = a_{35}^2 a_{53}$
$a_{35}^{a_5} = a_{35}^2 a_{45}^2 a_{53}$
$a_{35}^{a_{33}} = a_{35} a_{47}^2$
$a_{35}^{a_{34}} = a_{35} a_{48}^2$
$a_{36}^{a_1} = a_{36} a_{44}^2$
$a_{36}^{a_4} = a_{36}^2 a_{46} a_{53}^2$
$a_{36}^{a_5} = a_{36}^2 a_{53}^2$
$a_{36}^{a_{33}} = a_{36} a_{49}^2$
$a_{36}^{a_{34}} = a_{36} a_{50}^2$
$a_{37}^{a_2} = a_{37} a_{45}$
$a_{37}^{a_4} = a_{37}^2 a_{53}^2$
$a_{37}^{a_5} = a_{37} a_{44} a_{51}$
$a_{37}^{a_{32}} = a_{37} a_{47}$
$a_{37}^{a_{34}} = a_{37} a_{51}^2$
$a_{38}^{a_1} = a_{38} a_{45}^2$
$a_{38}^{a_4} = a_{38}^2$
$a_{38}^{a_5} = a_{38} a_{52}$
$a_{38}^{a_{32}} = a_{38} a_{49}$
$a_{38}^{a_{34}} = a_{38} a_{52}^2$
$a_{39}^{a_2} = a_{39} a_{46}$
$a_{39}^{a_4} = a_{39} a_{51}^2$
$a_{39}^{a_5} = a_{39}^2$
$a_{39}^{a_{32}} = a_{39} a_{48}$
$a_{39}^{a_{33}} = a_{39} a_{51}$
$a_{40}^{a_1} = a_{40} a_{46}^2$
$a_{40}^{a_4} = a_{40} a_{44}^2 a_{52}^2$
$a_{40}^{a_5} = a_{40}^2$
$a_{40}^{a_{32}} = a_{40} a_{50}$
$a_{40}^{a_{33}} = a_{40} a_{52}$
$a_{41}^{a_1} = a_{41} a_{47}$
$a_{41}^{a_2} = a_{41} a_{49}$
$a_{41}^{a_4} = a_{41} a_{51}^2$
$a_{41}^{a_5} = a_{41}^2 a_{53}^2$
$a_{41}^{a_{34}} = a_{41} a_{53}^2$
$a_{42}^{a_1} = a_{42} a_{48}$
$a_{42}^{a_2} = a_{42} a_{50}$
$a_{42}^{a_4} = a_{42}^2 a_{53}$
$a_{42}^{a_5} = a_{42} a_{52}$
$a_{42}^{a_{33}} = a_{42} a_{53}$
$a_{43}^{a_1} = a_{43} a_{51}$
$a_{43}^{a_2} = a_{43} a_{52}$
$a_{43}^{a_4} = a_{43}^2 a_{47}$
$a_{43}^{a_5} = a_{43}^2 a_{50}^2$
$a_{43}^{a_{32}} = a_{43} a_{53}^2$
$a_{44}^{a_4} = a_{44}^2$
$a_{44}^{a_5} = a_{44}^2$
$a_{45}^{a_4} = a_{45}^2$
$a_{46}^{a_5} = a_{46}^2$
$a_{47}^{a_5} = a_{47}^2$
$a_{48}^{a_4} = a_{48}^2$
$a_{49}^{a_5} = a_{49}^2$
$a_{50}^{a_4} = a_{50}^2$
$a_{51}^{a_4} = a_{51}^2$
$a_{51}^{a_5} = a_{51}^2$
$a_{52}^{a_4} = a_{52}^2$
$a_{52}^{a_5} = a_{52}^2$

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