COMPUTING IN GROUPS WITH EXPONENT SIX

GEORGE HAVAS, M. F. NEWMAN, ALICE C. NIEMEYER
AND CHARLES C. SIMS

Abstract. We have investigated the nature of sixth power relations required to provide proofs of finiteness for some two-generator groups with exponent six. We have solved various questions about such groups using substantial computations. In this paper we elaborate on some of the calculations and address related problems for some three-generator groups with exponent six.

1. Introduction

Motivated by an aim to get estimates for the number and length of sixth power relations which suffice to define groups with exponent six, we studied finiteness proofs for presentations of such groups in [5]. We tried to find relatively small sets of defining relations for various groups, with a view to improving our understanding of finiteness proofs.

We denote the free group on d generators with exponent n by $B(d, n)$ and generally use notation as in [5]. One question we would very much like to be able to answer is whether $B(2, 6)$ can be defined without using too many sixth powers. Here we focus on the computational components of the process, giving sample code which solves some associated problems.

We showed that $B(2, 6)$ has a presentation on 2 generators with 81 relations, which is derived from a polycyclic presentation. Here in Section 3 we give a program to construct a polycyclic presentation for $B(2, 6)$ which shows the structure of the group. If only sixth power relations are used, we showed that M. Hall's finiteness proof [2] yields that fewer than 2^{124} sixth powers can define $B(2, 6)$ [5, Theorem 2]. On the other hand the best lower bound we have proved is that at least 22 sixth powers are needed [5, Theorem 1]. We expect that 22 is closer to the truth than 2^{124}. We observed that current proof methods cannot yield fewer than 2^{95} sixth powers in a defining set of relators.

It seems unlikely that the computer-based methods now available — coset enumeration and Knuth-Bendix string rewriting — will succeed in finding

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'small' defining sets of sixth powers for $B(2,6)$ in the foreseeable future. Because of this we considered presentations related to some quotients of $B(2,6)$ for which coset enumeration and rewriting methods enable progress to be made. In a sense this is what Hall did. For him a major stepping stone was a lemma about a three-generator group with exponent six which proved to be finite of order 54 [2, Lemma 4]. He described this as the hardest lemma in his proof [3, p. 338]. It is easy to prove using coset enumeration or string rewriting.

We considered group presentations $\{X \mid \mathcal{R}\}$ with $X, \mathcal{R}$ finite to which the exponent six condition is added; we write $\{X \mid \mathcal{R}, \exp 6\}$. The groups defined in this way are finite and so a finite set of sixth power relators suffices to enforce the exponent six condition. Thus we studied presentations $\{X \mid \mathcal{R}, \mathcal{S}\}$ where $\mathcal{S}$ is a finite set of sixth powers and in some interesting cases found small sets $\mathcal{S}$ for which the groups $\langle X \mid \mathcal{R}, \mathcal{S}\rangle$ and $\langle X \mid \mathcal{R}, \exp 6\rangle$ defined by the presentations are isomorphic.

Let $C(r,s)$ denote the largest two-generator group with exponent six generated by elements of orders r and s. To understand $B(2,6)$ better we looked at presentations for the groups $C(2,2)$, $C(2,3)$, $C(3,3)$, $C(2,6)$ and $C(3,6)$. Let $\{a,b\}$ be a generating set for $B(2,6)$. The subgroup $H = \langle a^{6/r}, b^{6/s}\rangle$ of $B(2,6)$ is clearly a quotient group of $C(r,s)$. It turns out that H and $C(r,s)$ are isomorphic. The order of H and the index of the normal closure in $B(2,6)$ of $\langle a^r, b^s\rangle$ are easily computed using a polycyclic presentation for $B(2,6)$. In each case these numbers are the same. Programs for some of this are given in Section 3.

The presentation $\{a, b \mid a^2, b^2, (ab)^6\}$ is a minimal presentation for the group $C(2,2)$, the dihedral group of order 12. A minimal presentation for $C(2,3)$, a group of order 216, is $\{a, b \mid a^2, b^3, (ab)^6, [a,b]^6\}$. Therefore the first challenging case to consider is the group $C(3,3)$. It is a group of order $2^{10}3^3$. It turns out to be quite manageable. In this paper we concentrate on computational issues; the main matters can be explained in the context of $C(3,3)$ (Section 4), so we restrict attention to that case. However, we also have some results for $C(2,6)$ and $C(3,6)$ in [5].

In Section 5 we look at the three-generator group which is the key to Hall's finiteness proof. We finish (Section 6) by providing new results about presentations for another three-generator group with exponent six. The group is the relatively free product of a symmetric group on three symbols and a group of order 3; surprisingly it embeds in $B(2,6)$.

2. Computational tools

The major computational tools we use are outlined in [8]. We exhibit current capabilities of various computational methods and illustrate the use of a powerful new tool, a soluble quotient program, see Niemeyer [11]. Coset enumeration can now routinely handle hundreds of millions of cosets; the implementation used is the one by Havas and Ramsay [6], based on the description in [4]. For rewriting (see [13, Chapter 2]) we use the Rutgers Knuth-Bendix Package (RKBP) written by Sims [14]. Such programs are available as standalone packages [8] and in systems such as GAP [12], MAGMA [1] and **quotpic** [9]. We provide some programs, together with input files for and outputs from these systems.

We present real code used at the time of the Conference to solve the given problems. Thus we give a snapshot of capabilities as at July 1998. With continuing improvements in systems such as GAP and MAGMA we would expect these problems could be addressed entirely within one system.

3. A presentation for $B(2, 6)$

In [5] we gave a polycyclic presentation for $B(2, 6)$ based on a composition series with 53 generators and 1431 relations; this presentation shows much of the group's structure. The presentation is consistent and was obtained using the ANU Soluble Quotient Program (ANU SQ) [11] and GAP. Here we detail a GAP program which computes this presentation. The basic approach is the same as that used by Stephen Glasby, the first person to obtain a polycyclic presentation for $B(2, 6)$. We wanted a presentation which exhibits the structure of the group in a readily visible form, and this needs more effort.

Let F be the free group of rank 2 on $\{a, b\}$. Then F has a normal subgroup F^3 which is the subgroup generated by all cubes in F. Since F/F^3 has order 27, it follows from Schreier's Theorem that F^3 is free of rank 28 and hence F^3/M, where $M = (F^3)^2$, is an elementary abelian 2-group of order 2^{28}. Let G_1 denote F/M. Then G_1 has order $2^{28}3^3$ and has exponent six. Let G_2 denote F/L, where $L = (F^2)^3$. Then G_2 has order $2^2 3^{25}$ and exponent six. The group $B(2, 6)$ is isomorphic to $F/(M \cap L)$ (see [5]). Moreover, $F/(M \cap L)$ embeds into the direct product $F/M \times F/L$. Clearly $F/M \times F/L$ is a group with exponent six. The two-generator subgroup of it generated by (aM, aL) and (bM, bL) is isomorphic to $B(2, 6)$.

Using ANU SQ we computed consistent power conjugate presentations for the groups G_1 and G_2 and epimorphisms $\kappa : F \to G_1$ and $\lambda : F \to G_2$. This is straightforward. We use enough sixth powers to ensure that the computed quotients are isomorphic to G_1 and G_2.

```
##   Use the  ANU SQ  package to compute  consistent power conjugate
##   presentations  for  the  groups   G1  and   G2   and  epimorphisms
##   lambda: F -> G1   and   kappa: F -> G2   where F is free of rank 2
##
RequirePackage("anusq");
F := FreeGroup("a", "b");; a:=F.1;; b:=F.2;;
G1 := AgGroupFpGroup( Sq(
           F / [ a^6, b^6 ],
           [ [3, 2], # 2 steps in the exponent-3 series
             [2, 1]  # 1 step in the exponent-2 series
             ] ));
```

```
lambda := GroupHomomorphismByImages( F, G1, [F.1, F.2], [G1.1, G1.2] );

a:=F.1;;   # Needed because Sq redefines a
G2 := AgGroupFpGroup( Sq(
           F / [ a^6, b^6, (a*b)^6, (a*b^-1)^6, (a^2*b^2)^6, (a^2*b^-2)^6,
                 (a*b*a*b^-1)^6, (a*b*a^-1*b)^6, (a*b*a^-1*b^-1)^6,
                 (a^3*b*a*b)^6 , (a^-2*b^2*a*b*a*b)^6 ],
           [ [2, 1], # 1 step in the exponent-2 series
             [3, 3]  # 3 steps in the exponent-3 series
             ] ) );
kappa := GroupHomomorphismByImages( F, G2, [F.1, F.2], [G2.1, G2.2] );
```

It is not too difficult to obtain a presentation for $B(2,6)$ from here. However it is more challenging to find one which shows much of the group's structure. The GAP code in Appendix A computes such a presentation. It uses two auxiliary functions `Action` (Appendix B) and `BaseChangeAgGroup` (Appendix C). Function `Action` rewrites the conjugation action of the generators of a section of a group on an elementary abelian p-subgroup as matrices over $GF(p)$. Function `BaseChangeAgGroup` computes an automorphism of a given soluble group mapping the GAP-computed polycyclic presentation to one based on a user supplied polycyclic generating set.

We start in Appendix A by computing `I`, a special polycyclic presentation for $B(2,6)$ inside the direct product, and `chi`, a homomorphism from the free group on 2 generators to `I`. Next we compute matrices for the action of generators of $B(2,3)$ on the elementary abelian group of order 2^{28}. We then completely reduce the module into two trivial, four 2-dimensional and three 6-dimensional submodules.

Next we compute the images in the direct product $G_1 \times G_2$ of generating sets of these submodules. Then, in a similar fashion, we act with the 2 generators of order 2 on an elementary abelian section of order 3^5 and compute the images in $G_1 \times G_2$ of the generators of the submodules in the decomposition of this module. Finally, this allows us to define a new polycyclic presentation for $B(2,6)$ exhibiting this structure. With the GAP function `BaseChangeAgGroup` we compute a homomorphism `chi` from the free group on a and b into $B(2,6)$.

We are able to compute in such presentations for $B(2,6)$ quite readily. For example, we can compute the order of a subgroup mentioned in Section 1 as follows.

```
b1 := Image( chi, chi.source.1 );
b2 := Image( chi, chi.source.2 );
C26 := Subgroup( B26, [ b1^3, b2 ] );
# Order of C(2,6) is
Size(C26);
```

4. The group $C(3,3)$

As pointed out in the introduction, the first challenging quotient of $B(2,6)$ obtained by applying order conditions on the generators is the factor group $C(3,3)$ given by the presentation $\{a, b \mid a^3, b^3, \exp 6\}$. One effective way of studying this group is to use **quotpic**, which provides useful pictures of groups and their quotients.

The order of $C(3,3)$ can easily be computed using finiteness and the fact [3, Theorem 18.4.6] that the 2-length and the 3-length are 1. From this it follows that a Sylow 3-subgroup of $C(3,3)$ is isomorphic to the Sylow 3-subgroup of $(C(3,3)^2)^3$ and the Sylow 2-subgroup to that of $(C(3,3)^3)^2$. Since $C(3,3)^2 = C(3,3)$, a Sylow 3-subgroup is (isomorphic to) $B(2,3)$ and has order 27.

Give **quotpic** the presentation $\{a, b \mid a^3, b^3\}$. Let F be the group this presentation defines. Then F^3 is the kernel of the largest 3-quotient of class 2. The kernel is free of rank 10. Hence $F/(F^3)^2$ has order 27×2^{10}. Therefore the order of $C(3,3)$ is 27×2^{10}. Using **quotpic** to calculate the intermediate quotients between F^3 and $(F^3)^2$ shows (Figure 1) that $C(3,3)^3$ is the direct sum of three irreducible $B(2,3)$-modules with dimensions 2, 2 and 6.

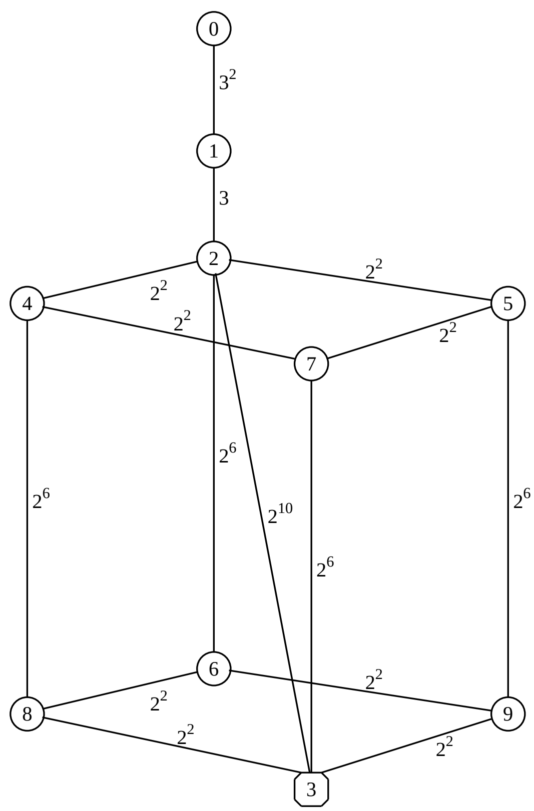

Figure 1. F^3 and $(F^3)^2$.

The diagram in Figure 1 is produced by **quotpic** (after judicious movement of vertices), where vertex 2 represents F^3 and vertex 3 represents $(F^3)^2$. The primary interface to **quotpic** is graphical and hard to describe in words. Suffice it to say, the input used to produce this diagram comprised a file with 22 non-whitespace characters plus a small number of mouse clicks (less than a dozen if vertex movement is not counted).

We showed [5, Theorem 9] that every presentation with the two relators a^3 and b^3 and otherwise sixth powers needs at least seven relators. The proof consists of a careful analysis of quotients of the group $F = \langle a, b \mid a^3, b^3\rangle$ which appear in Figure 2 (which extends the part of Figure 1 without the intermediate quotients).

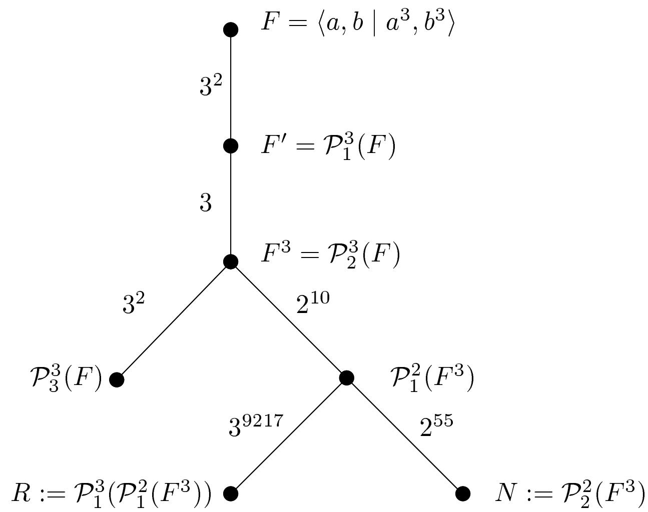

FIGURE 2. Quotients of F

Methods similar to those in the proof allowed us to use a polycyclic presentation for $Q := F/N$, whose order is $3^3 2^{65}$, to find short presentations $\{a, b \mid a^3, b^3, w_1^6, \dots, w_n^6\}$ which define groups whose largest meta-nilpotent quotient is $C(3,3)$. For such groups, we check that the largest soluble quotient is also $C(3,3)$. Having found candidate presentations with the correct soluble quotient, it remains to see if we can prove that they actually present $C(3,3)$. For brevity in the following we sometimes use the case inverse convention in which A and B denote a^{-1} and b^{-1}, respectively.

In [5] we use many applications of coset enumeration and Knuth-Bendix rewriting to prove various results for the groups $C(3,3)$, $C(2,6)$ and $C(3,6)$. Here we provide details on one application of each method to $C(3,3)$. All successful coset enumerations and Knuth-Bendix rewritings reported in [5] have been done in similar ways.

Let $\mathcal{S}_1 = \{(ab)^6, (aB)^6, (abaB)^6, (abAb)^6, (abAB)^6, (ababAB)^6, (abaBAb)^6, (abaBAB)^6, (abAbaB)^6, (abAbAB)^6, (abABaB)^6\}$. This includes one representative of all words of length up to six which are different as potential relators for $C(3,3)$. We proved $\Pi_1 = \{a, b \mid a^3, b^3, \mathcal{S}_1\}$ is a presentation for $C(3,3)$ by coset enumeration and by rewriting. The following MAGMA program does coset enumeration over the subgroup $\langle a, (ab)^3 \rangle$ and prints out the index plus some statistics. The performance of coset enumeration depends very much on the sequence of coset definitions. The values of the parameters `Strategy` and `SubgroupRelations` which appear were obtained by guided experimentation. These parameters control the sequence of coset definitions and yield the results quoted here. For further details on coset enumeration parameters see [7].

```
C33<a,b> := Group< a, b | a^3, b^3, (a*b)^6, (a*b^-1)^6, (a*b*a*b^-1)^6,
            (a*b*a^-1*b)^6, (a*b*a^-1*b^-1)^6, (a*b*a*b*a^-1*b^-1)^6,
            (a*b*a*b^-1*a^-1*b)^6, (a*b*a*b^-1*a^-1*b^-1)^6,
            (a*b*a^-1*b*a*b^-1)^6, (a*b*a^-1*b*a^-1*b^-1)^6,
            (a*b*a^-1*b^-1*a*b^-1)^6 >;
H := sub< C33 | a, (a*b)^3 >;
time I,CT,M,T := ToddCoxeter(C33, H : CosetLimit:=5000000,
                    Strategy:=<4300,5>, SubgroupRelations:=2 );
print I, M, T;
```

In less than four minutes on a moderate speed workstation we obtain index 1152 with a maximum of 3245801 and total of 3417675 cosets defined. Together with a theoretical argument given in [5], this shows that Π_1 presents $C(3,3)$.

Now consider $\mathcal{S}_2 = \{(ab)^6, (aB)^6, (abaB)^6, (abAb)^6, (abAB)^6, (abaBAb)^6, (aBaBAb)^6, (aBabAB)^6\}$. This yields a ten-relator presentation for $C(3,3)$ which includes 8 sixth powers. By applying RKBP to $\{a, b \mid a^3, b^3, \mathcal{S}_2\}$ we obtain a confluent presentation from which we can compute the group order and much other information. The confluent rewriting system has 5016 rules with longest left and right hand sides of length 22. This calculation is much harder than for $\mathcal{S}_1$, taking about 200 cpu minutes on a moderate speed PC. The following RKBP input together with three auxiliary files suffices to do the computation.

```
echo
input c33
!cat c33.rl1
summary
kb 28 -1 28 28
summary
quit
```

Suitable auxiliary files are shown next in columns headed by file names.

c33.rl1	c33.sb1	c33.sys
`3 a^3 0 @`	`subsystem`	`system`
`3 b^3 0 @`	`rule_type:  subword`	`word_type:  group`
`12 (ab)^6 0 @`	`index_type:  automaton`	`ngens:  2`
`12 (aB)^6 0 @`	`external:  len_names`	`naming_convention:  none`
`24 (abaB)^6 0 @`	`compressed:  no`	`numbering_convention:  none`
`24 (abAb)^6 0 @`	`rule_file:  c33.rl1`	`inverse_convention:  none`
`24 (abAB)^6 0 @`		`generators: a A b B`
`36 (abaBAb)^6 0 @`		`inverse_pairs: a  A b  B`
`36 (aBaBAb)^6 0 @`		`weight_convention:  constant`
`36 (aBabAB)^6 0 @`		`level_convention:  constant`
		`nsub:  1`
		`subsystems:  c33.sb1`

5. A group of order 54

Hall [2, p. 771] wrote:

> *If $H = \{x, a, b\}$ is of exponent six, and if $x^2 = 1$, $a^3 = 1$, $b^3 = 1$, $xax = a^{-1}$, $xbx = b^{-1}$, then $\{a, b\}$ is of exponent three.*
>
> This lemma is critical since we note that $[H : H'] = 2$, and so if H is finite, then $H' = \{a, b\}$ must be of exponent three and so of order 27 (or naturally a divisor of 27). Thus if H is finite, its order divides 54.

Hall takes over three pages to prove the lemma, which is readily proved by coset enumeration using four sixth powers (see [10]). A Magma program (including an implicit coset enumeration) which shows that this group has order 54 is trivial:

```
Hall54<x,a,b> := Group< x, a, b | x^2, a^3, b^3, (x*a)^2, (x*b)^2,
                   (a*b)^6, (a*b^-1)^6, (x*a*b)^6, (x*(a*b)^3)^6 >;
print Order(Hall54);
```

Following one theme, we ask how few sixth powers are required in this context. It is straightforward to show that at least three are required using a module-based argument. Can it be done with three?

Arguments like those in [5, Proof of Theorem 9] tell us that if it can be done with three sixth powers they will have the forms $(abu)^6$, $(aBv)^6$ and $(xw)^6$, where both u and v are in the derived subgroup of $\langle a, b\rangle$ and w in $\langle a, b\rangle$. For simplicity we select u and v trivial and consider possibilities for w.

Let $\Pi_3 = \{\ x, a, b \mid x^2, a^3, b^3, (xa)^2, (xb)^2, (ab)^6, (aB)^6, (xw)^6\ \}$. It is generally easier to experiment with two-generator groups rather than with three-generator groups. In this case the subgroup $\langle a, b\rangle$ of $\langle \Pi_3\rangle$ has index 2, so we study it. Reidemeister-Schreier rewriting shows that this subgroup has the presentation $\Pi_4 = \{\ a, b \mid a^3, b^3, (ab)^6, (aB)^6, (w\bar{w})^3\ \}$, where $\bar{w}$ is obtained from w by replacing each symbol in the word w by its inverse. Further Π_3 presents the required group if and only if Π_4 is a presentation for $B(2, 3)$.

We can readily test whether there are candidate presentations of the form Π_4. We consider words w in the free product F of two groups of order 3 defined by $\{a, b \mid a^3, b^3\}$. We construct words as alternating products of $a^{\pm 1}$ and $b^{\pm 1}$, normalised to start with a.

For all such words w of length less than eight we find that the largest soluble quotient of $\langle \Pi_4 \rangle$ is not $B(2,3)$. At length eight we find 16 words $w_1, \ldots, w_{16}$ which do yield the required soluble quotient. The following simple Magma code suffices for this purpose.

```
F<a,b> := FreeGroup(2);
for x2, x4, x6, x8 in [b, b^-1], x3, x5, x7 in [a, a^-1] do
  relns := [ a^3, b^3, (a*b)^6, (a*b^-1)^6,
    (a*x2*x3*x4*x5*x6*x7*x8*(x8*x7*x6*x5*x4*x3*x2*a)^-1)^3 ];
  G := quo< F | relns >;
  Q := pQuotient (G, 3, 3: Print := 0);
  if Order (Q) eq 27 then
    D := ncl < G | (G.2, G.1, G.1), (G.2, G.1, G.1) >;
    x := AbelianQuotientInvariants (D);
    if x eq [] then print "POSSIBLE", G; end if;
  end if;
end for;
```

The 16 words w_i that we obtain are: $abaBAbab$, $abaBABab$, $abAbaBab$, $abAbaBAb$, $abABabaB$, $abABaBAb$, $abABAbab$, $abABAbaB$, $aBabAbaB$, $aBabABaB$, $aBAbabAB$, $aBAbaBab$, $aBAbABab$, $aBAbABaB$, $aBABabaB$, $aBABabAB$. Some of the words $w_i\bar{w}_i$ are conjugate in the free group, meaning that we do not get 16 "essentially different" presentations. However, since coset enumeration can handle presentations with even minor perturbations very differently we chose to consider them all.

We are unable to determine whether any of the groups $\langle \Pi_4 \rangle$ derived from these words is finite. It is interesting that four of the words $w_i\bar{w}_i$ can be written as left-normed commutators of length four. Thus from w_7, w_8, w_{13} and w_{14} we obtain $[a, B, A, B]$, $[a, B, A, b]$, $[a, b, A, B]$ and $[a, b, A, b]$, respectively. In a sense this explains why proving finiteness (should that be the case) is hard. We would need to derive the short relators $(ab)^3$ and $(aB)^3$ which hold in $B(2,3)$ from the initial four relators together with a relator like $[a, B, A, B]^3$.

6. Another three-generator group

Now that we have looked at one three-generator group with exponent six we consider another which also arises in a natural way and which may contribute to further study of presentations for $B(2,6)$. It is the relatively free product of the symmetric group S_3 on three letters and a group of order 3; we call it $C(S_3, Z_3)$. It has presentation $\Pi_5 = \{a, b, c \mid a^2, b^3, (ab)^2, c^3, \exp 6\}$, and its order is 17496.

It is not immediately obvious that this group embeds in $B(2,6)$ but a few random choices produce the following example. Let x and y be generators for $B(2,6)$. Then $\langle x^3, (x^3y^3)^2 \rangle$ is an S_3 and with $(xyx)^2$ it generates a subgroup

H which is a quotient of $C(S_3, Z_3)$. We compute that H has order 17496, so $C(S_3, Z_3)$ embeds in $B(2, 6)$.

We now show how to find a small set of sixth powers instead of the exponent six condition in Π_5 to present $C(S_3, Z_3)$. Our process makes use of subtle relations which hold in $C(3, 3)$. A relation is *subtle* if it is shorter than the longest sixth power required for a proof that the relation holds in a suitable context (for more on subtlety see [5]).

Let $G = \langle \Pi_5 \rangle$. Then $\langle b, c \rangle$ is a quotient of $C(3, 3)$. Let $u = (bc)^3$, so $u^2 = 1$. From our knowledge of $C(3, 3)$, we can see that $[b, u]^2 = 1$. An easy enough computation (coset enumeration or Knuth-Bendix) shows that the relations $a^2 = b^3 = (ab)^2 = u^2 = [b, u]^2 = 1$ plus the sixth powers of au, abu, aBu, and $aubu$ imply that $[b, u] = 1$. Thus in G we have $[b, (bc)^3] = 1$. By symmetry, we have $[b, (bC)^3] = 1$ too. Adding these two relations to $C(3, 3)$ gives a group of order 27 in which $(bc)^3 = (bC)^3 = 1$. Once we have these relations, it does not take many sixth powers to get the order of G. Certainly bases of length at most 5 suffice.

Following this proof, but without explicitly requiring the "very subtle" relators $(bc)^3$ and $(bC)^3$ (whose derivation from sixth powers is relatively hard) coset enumeration over the trivial subgroup for the presentation $\{a, b, c, u, v \mid a^2,\ b^3,\ c^3,\ (ab)^2,\ u = (bc)^3,\ v = (bC)^3,\ (bc)^6,\ (bC)^6,\ ([c, b][C, b])^2,\ ([b, c][B, c])^2,\ [b, u],\ (au)^6,\ (abu)^6,\ (aBu)^6,\ (aubu)^6,\ [b, v],\ (av)^6,\ (abv)^6,\ (aBv)^6,\ (avbv)^6,\ (ac)^6,\ [c, a]^6,\ (abc)^6,\ (aBc)^6,\ (abcac)^6,\ (abCac)^6,\ (aBcaC)^6\}$ gives index 17496 defining a maximum of 860646 and a total of 1559920 cosets. Here $([c, b][C, b])^2$ and $([b, c][B, c])^2$ are subtle relators from $C(3, 3)$.

Deleting any one of the three base length 5 sixth powers gives a group which is 3 times bigger.

Having proved finiteness this way we can prune the proof by using the result of a different enumeration. The index of $\langle a, c \rangle$ in the group with presentation $\{a, b, c, u, v \mid a^2,\ u^2,\ v^2,\ b^3,\ c^3,\ (ab)^2,\ u = (bc)^3,\ v = (bC)^3,\ (bc)^6,\ (bC)^6,\ [b, u],\ [b, v],\ (ac)^6,\ [c, a]^6,\ (abc)^6,\ (aBc)^6,\ (abcac)^6,\ (abCac)^6,\ (aBcaC)^6\}$ is readily shown to be 81, from which finiteness again follows. Suffice it to say that this leads to a presentation for $C(S_3, Z_3)$ of the form Π_5 with a reasonable number of sixth powers for which we can prove finiteness.

Perhaps the next sensible three-generator problem to consider is short presentations for $C(V_4, Z_3) = \{a, b, c \mid a^2, b^2, (ab)^2, c^3, \exp 6\}$. Investigations so far show this to be more difficult.

References

[1] Wieb Bosma, John Cannon and Catherine Playoust. The Magma algebra system I: the user language. *J. Symbolic Comput.* **24** (1997) 235–265.

[2] Marshall Hall Jr. Solution of the Burnside problem for exponent six. *Illinois J. Math.* **2** (1958) 764–786.

[3] Marshall Hall Jr. *The theory of groups*. The Macmillan Co., New York (1959).

[4] George Havas. Coset enumeration strategies. In *ISSAC'91, Proceedings of the* 1991 *International Symposium on Symbolic and Algebraic Computation.* ACM Press, New York (1991) 191–199.

[5] George Havas, M. F. Newman, Alice C. Niemeyer and Charles C. Sims. Groups with exponent six. *Commun. Algebra* **27** (1999) 3619–3638.

[6] George Havas and Colin Ramsay, *Coset enumeration: ACE version* 3 (1999). Available as `http://www.csee.uq.edu.au/~havas/ace3.tar.gz`.

[7] George Havas and Colin Ramsay. *Experiments in coset enumeration.* Technical Report 13, Centre for Discrete Mathematics and Computing, The University of Queensland, 1999.

[8] George Havas and Edmund F. Robertson. Application of computational tools for finitely presented groups. In *Computational support for discrete mathematics*, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. **15** (1994) 29–39.

[9] Derek F. Holt and Sarah Rees. A graphics system for displaying finite quotients of finitely presented groups. In *Groups and computation*, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. **11** (1993) 113-126.

[10] M. F. Newman. Groups of exponent six. In *Computational group theory.* Academic Press, London (1984) 39–41.

[11] Alice C. Niemeyer. A finite soluble quotient algorithm. *J. Symbolic Comput.* **18** (1994) 541–561.

[12] Martin Schönert et al. GAP – *Groups, Algorithms and Programming.* Lehrstuhl D für Mathematik, RWTH Aachen (`ftp://www.gap.dcs.st-and.ac.uk/gap`), fifth edition (1995).

[13] Charles C. Sims. *Computation with finitely presented groups*, Encyclopedia of Mathematics and its Applications **48**. Cambridge University Press, Cambridge (1994).

[14] Charles C. Sims. RKBP, Version 1.3 (1995). Available via anonymous ftp from `dimacs.rutgers.edu`, in directory `/pub/rkbp`.

Appendix A. Compute $B(2,6)$

```
##   Compute the direct product G1xG2 of G1 and G2 (already computed)
##   and identify the image of B(2,6) in it.
F.relators := [];
G1xG2 := DirectProduct( G1, G2 );
em1 := G1xG2.embeddings[1]; em2 := G1xG2.embeddings[2];

delta1 := lambda*em1; delta2 := kappa*em2;
mu := GroupHomomorphismByImages( F, G1xG2, F.generators, [
  Image(delta1,F.1) * Image(delta2,F.1),
  Image(delta1,F.2) * Image(delta2,F.2)] );
immu := Image(mu, F); I := SpecialAgGroup(immu);
chi :=  mu.operations.CompositionMapping( I.bijection^-1, mu );
##  Now  I  is a special ag presentation for  B(2,6)  inside the
##  direct product  G1xG2  and  chi: F -> I  is a homomorphism.
##  Compute matrices for the action of the generators of B(2,3)
##  on the elementary abelian group of order 2^28.
##  Then determine the complete reduction of the module into:
##  T T 2 2 2 6 6 6.
acton := [I.1, I.2];
Append( acton, Sublist( I.generators, [6..31] ) );
mats1 := Action( I, acton, [I.3, I.4, I.5], GF(2) );
```

```
RequirePackage("matrix");
gmod1  := GModule( mats1 ); cp1 := CompositionFactors( gmod1 );

submods := [];
for i in [ 1 .. Length(cp1) ] do
    minsubs := MinimalSubGModules ( cp1[i][1], gmod1, 21 );
    span := []; d := 0;
    for j in [1 .. Length(minsubs)] do
        vs := VectorSpace( Union( span, minsubs[j]), GF(2) );
        if Dimension(vs) > d then
            d := Dimension(vs);
            span := Union( span, minsubs[j] );
            Add( submods, minsubs[j] );
        fi;
    od;
od;
newbI := []; newbI[1] := 0; newbI[2] := 0; newbI[3] := 0; s := 3;
for i in [ 1 .. Length(submods) ] do
    for j in [ 1.. Length(submods[i]) ] do
        x := I.1^Int(submods[i][j][1]) * I.2^Int(submods[i][j][2]);
        x := x * Product(List([3..Length(submods[i][j])],
                x->I.generators[3+x]^Int(submods[i][j][x])) )^3;
        Add(newbI, x );
    od;
od;
two1 := newbI[4]; two2 := newbI[5];
Print("#I mapped the module generators into the direct product \n");

##  Now determine the structure of the module where B(2,6)/B(2,6)^2
##  acts on the largest elementary abelian 3-quotient of B(2,6)^2;
##  it has order 3^5.  This is then a module for a group of order 4
##  and decomposes into:  T T ma mb mab,  where a fixes the
##  generator of ma,  b  fixes the generator of  mb  and ab fixes the
##  generator of mab (a, b the generators of the group of order 4).

mats2 := Action(I, [I.3, I.4, I.32, I.33, I.34 ], [I.1, I.2], GF(3));
gmod2 := GModule(mats2); cp2 := CompositionFactors( gmod2 ); i := 1;
while cp2[i][2] <> 2 do
    i := i + 1;
od;
Swap( cp2, 1, i);
Print("#I chopped the 5-dimensional module over GF(3) \n");

submods2 := [];
for i in [ 1 .. Length(cp2) ] do
    Add( submods2, MinimalSubGModules( cp2[i][1], gmod2, cp2[i][2] ));
od;

nbs := [];
for i in [ 1 .. Length(submods2) ] do
    for j in [ 1.. Length(submods2[i]) ] do
```

```
        for k in [ 1 .. Length(submods2[i][j]) ] do
               x := I.3^Int(submods2[i][j][k][1])*
                    I.4^Int(submods2[i][j][k][2])*
                    I.32^Int(submods2[i][j][k][3])*
                    I.33^Int(submods2[i][j][k][4])*
                    I.34^Int(submods2[i][j][k][5]);
            Add(nbs, x );
        od;
    od;
od;
Print("#I mapped the module generators into the direct product \n");

# Compute the exponents of the basis elements in the quotient group
# and then lift the elements into the big group.
u1 := nbs[1]; u2 := nbs[2]; u3 := Comm( u2, u1 );
# the 9-dimensional part
u4 := nbs[3];      u5 := nbs[4];      u6 := nbs[5];
u7 := Comm(u4,u1); u8 := Comm(u4,u2); u9 := Comm(u5,u1);
u10:= Comm(u5,u2); u11:= Comm(u6,u1); u12:= Comm(u6,u2);
# the 12-dimensional part
u13 := Comm(u5,u4); u14 := Comm(u6,u4); u15 := Comm(u6,u5);
u16 := Comm(u4,u3); u17 := Comm(u5,u3); u18 := Comm(u6,u3);
u19 := Comm(u7,u5); u20 := Comm(u7,u6); u21 := Comm(u8,u5);
u22 := Comm(u8,u6); u23 := Comm(u9,u6); u24 := Comm(u10,u6);
# the 1-dimensional part
u25 :=  Comm( Comm( u6, u5), u4 );

newbI[1] := u1; newbI[2] := u2; newbI[3] := u3;
Add(newbI,u4^2);  Add(newbI,u5^2);  Add(newbI,u6^2);  Add(newbI,u7^2);
Add(newbI,u8^2);  Add(newbI,u9^2);  Add(newbI,u10^2); Add(newbI,u11^2);
Add(newbI,u12^2); Add(newbI,u13^2); Add(newbI,u14^2); Add(newbI,u15^2);
Add(newbI,u16^2); Add(newbI,u17^2); Add(newbI,u18^2); Add(newbI,u19^2);
Add(newbI,u20^2); Add(newbI,u21^2); Add(newbI,u22^2);
Add(newbI,u23^2); Add(newbI,u24^2); Add(newbI,u25^2);
Print("#I computed a new basis for B(2,6) inside the direct product \n");

alpha := BaseChangeAgGroup( I, newbI );
B26 := alpha.range; chi := chi*alpha;

newb:=[];
newb[1] := B26.1*B26.41^2*B26.42^2*B26.47^2*B26.48*B26.50*B26.51^2;
newb[2] := B26.2*B26.41*B26.42^2*B26.43^2*B26.47*B26.49*B26.51^2*B26.52;
newb[3] := B26.3*B26.47*B26.48^2*B26.49*B26.50*B26.51^2;
for i in [ 4 .. Length(B26.generators)] do
    newb[i] := B26.generators[i];
od;

## Finally, determine the desired presentation for B26,
## and chi a homomorphism from F to B26.
beta := BaseChangeAgGroup( B26, newb);
B26 := beta.range; chi := chi * beta;
```

Appendix B. GAP function Action

```
Action := function(grp, gens, actgens, F)
    local  x, y, pos, e, mt, act;

    pos := List( gens, i->Position(grp.generators,i));
    act := [];
    for x in actgens do
        mt := [];
        for y in gens do
           e := Exponents(grp, y^x, F);
           Add(mt, List(pos, i->e[i]));
        od;
        Add( act, mt );
    od;

    return act;
end;
```

Appendix C. GAP function BaseChangeAgGroup

```
BaseChangeAgGroup := function( g, newb )
    local  ser, i, k, igs, u, alpha;

    ser := [];
    for i in [1..Length(newb)] do
        k := Subgroup( g, Sublist(newb,[i..Length(newb)]));
        Cgs( k );  Add( ser, k );
    od;
    Add( ser, TrivialSubgroup(g) );
    alpha := IsomorphismAgGroup( ser );  k := alpha.range;

    igs := List( newb, x -> Image(alpha,x) );
    u := Subgroup( k, igs );  u.igs := igs;
    u.operations.AddShiftInfo(u);
    g := AgGroupFpGroup( FpGroup(u) );
    alpha := GroupHomomorphismByImages( ser[1], g, newb, g.generators );

    return alpha;
end;
```

George Havas,
The University of Queensland,
Queensland 4072,
Australia.

M. F. Newman,
Australian National University,
Canberra 0200,
Australia.

Alice C. Niemeyer,
University of Western Australia,
Nedlands, WA 6907,
Australia.

Charles C. Sims,
Rutgers University,
New Brunswick, NJ 08903,
USA.