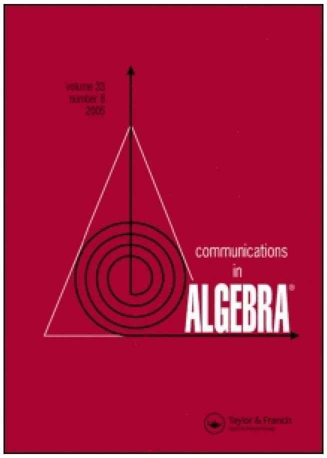

Communications in Algebra
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597239

CERTAIN CYCLICALLY PRESENTED GROUPS ARE INFINITE

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Online Publication Date: 30 September 2001
To cite this Article: Havas, George, Holt, Derek F. and Newman, M. F. (2001) 'CERTAIN CYCLICALLY PRESENTED GROUPS ARE INFINITE', Communications in Algebra, 29:11, 5175 - 5178
To link to this article: DOI: 10.1081/AGB-100106810
URL: http://dx.doi.org/10.1081/AGB-100106810

COMMUNICATIONS IN ALGEBRA, 29(11), 5175–5178 (2001)

CERTAIN CYCLICALLY PRESENTED GROUPS ARE INFINITE

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ABSTRACT

We prove that the groups in two infinite families considered by Johnson, Kim and O'Brien are almost all infinite.

1 INTRODUCTION

In a recent paper by Johnson, Kim and O'Brien [4], two pairs of infinite families $G_1(n), G_2(n)$ $(n \geq 1)$ and $H_1(n), H_2(n)$ $(n \geq 1)$ of cyclically presented groups are considered. The corresponding groups in each pair have the same Alexander polynomials, and it is proved that these corresponding groups are isomorphic in all cases. In Question 4.3 of that paper, the authors ask which of these groups are finite. It is clear from the manipulations in [4] that in the degenerate cases $n \leq 2$ the groups are finite and cyclic.

Here we show that they are infinite whenever $n \geq 3$.

This result is not surprising though it was not entirely straightforward to find; there is a brief description in Section 3. The justification, given in Section 2, is easy but somewhat different from familiar infiniteness proofs.

2 THE PROOF

Since it is shown in [4] that $G_1(n) \cong G_2(n)$ and $H_1(n) \cong H_2(n)$ for all n, we need only consider $G_1(n)$ and $H_1(n)$, which are defined by

$$G_1(n) = \langle x_i \mid x_{i+1}^{-1}x_{i+2}x_{i+1}^{-1}x_{i+2}x_i x_{i+1}^{-1}x_i\rangle,$$
$$H_1(n) = \langle x_i \mid x_{i+1}^{-1}x_{i+2}x_{i+1}^{-1}x_{i+2}x_{i+1}^{-2}x_i x_{i+1}^{-1}x_i\rangle,$$

where each group has n generators and n relators, which are obtained from the given one by applying powers of the cycle $(1, 2, \ldots, n)$ to the subscripts and reducing these modulo n to lie in the set $\{1, 2, \ldots, n\}$.

Each of these groups clearly has an automorphism of order n mapping x_i to x_{i+1} for each i, where the subscripts are again reduced modulo n. By taking the semidirect products of $G_1(n)$ and $H_1(n)$ with this automorphism, we obtain two-generator groups defined by:

$$EG_1(n) = \langle t, x \mid t^n, x^{-t}x^{t^2}x^{-t}x^{t^2}xx^{-t}x\rangle,$$
$$EH_1(n) = \langle t, x \mid t^n, x^{-t}x^{t^2}x^{-t}x^{t^2}x^{-2t}xx^{-t}x\rangle.$$

Now, for $n \geq 7$ we can construct a transitive permutation representation of $EG_1(n)$ on the set $\{\alpha_{ij} \mid i \in \mathbb{Z}, 1 \leq j \leq n\}$ as follows:

$$t \to \prod_{i\in\mathbb{Z}}(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}), \quad x \to \prod_{i\in\mathbb{Z}}(\alpha_{i1}, \alpha_{i+1,3}, \alpha_{i+2,5})(\alpha_{i+2,6}, \alpha_{i+1,4}, \alpha_{i2}).$$

It is routine to check that the images of this map satisfy the group relators, and hence that the map is a homomorphism for $n \geq 7$. (It is convenient to define $y = x^t$ and $z = y^t$ to perform the check.)

Since orbits of the image of $\langle t\rangle$ are all fused under the action of the image of x, the representation is transitive. Of course a group transitive on an infinite set is infinite. In fact xt^{-2} has infinite order.

Similarly, for $n \geq 10$, the following map defines a transitive permutation representation of $EH_1(n)$ on the same set:

$$t \to \prod_{i\in\mathbb{Z}}(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}), \quad x \to \prod_{i\in\mathbb{Z}}(\alpha_{i1}, \alpha_{i+1,4}, \alpha_{i+2,7})(\alpha_{i+2,9}, \alpha_{i+1,6}, \alpha_{i3}).$$

Here xt^{-3} has infinite order.

These general mappings do not work for smaller n. We can establish infiniteness in those cases using the now standard method of finding a subgroup N of finite index with infinite abelianization. In fact we have found a normal subgroup N with this property in all cases, and we list the structure of the quotients G/N in the tables below. In each case the chosen normal subgroup is maximal with respect to having infinite abelianization.

G	$\|G/N\|$	G/N
$G_1(3)$	336	$L_2(7)\cdot 2$
$G_1(4)$	3	C_3
$G_1(5)$	60	A_5
$G_1(6)$	168	$L_2(7)$
$H_1(3)$	660	$L_2(11)$
$H_1(4)$	168	$L_2(7)$
$H_1(5)$	3420	$L_2(19)$
$H_1(6)$	3	C_3
$H_1(7)$	1092	$L_2(13)$
$H_1(8)$	168	$L_2(7)$
$H_1(9)$	504	$L_2(8)$

3 OUR APPROACH

Since we expected most of the groups to be infinite we began by using MAGMA [2] and **quotpic** [3], to look for infinite sections. With programs like [1, H20E42], we considered abelian quotients of subgroups of low index and at the cores of such subgroups in the groups $G_1(n)$ and $H_1(n)$. By looking at $3 \le n \le 6$ and subgroups with single digit index we could observe infinite sections in all cases. We made a standard application of the Golod-Shafarevich theorem on 2-quotients of cores in the cases $G_1(3)$ and $H_1(5)$, with a program similar to [1, H20E38].

The low index subgroup algorithm depends exponentially on the number of group generators, and finding low index subgroups for $n \ge 7$ becomes very time consuming for n-generator groups. This was resolved by studying instead the two-generator extensions $EG_1(n)$ and $EH_1(n)$. In fact, using the `LowIndexProcess` command in MAGMA with the

`ColumnMajor parameter set true`, we could rapidly find low index subgroups in $EG_1(n)$ and $EH_1(n)$ with infinite abelian quotients for $7 \leq n \leq 100$. The pattern became regular in that there were such subgroups of index n for $n \geq 7$ in the $EG_1(n)$ case and for $n \geq 10$ in the $EH_1(n)$ case, and the image of x in the corresponding quotients was a product of two 3-cycles. To do all of these took less than 7 cpu minutes on a 266 MHz PC running the linux version of MAGMA. It was the study of these which led to the permutation representations in the previous section.

For the groups $G_1(n)$ with $3 \leq n \leq 6$ and the groups $H_1(n)$ with $3 \leq n \leq 9$ most of the subgroups tabulated in the previous section can be found as cores of low index subgroups produced using MAGMA. However in practice we first found some of them using **quotpic**.

ACKNOWLEDGMENTS

This work was undertaken while all three authors were at the Mathematics Institute to take part in the Warwick Workshop on Computation in Group Theory and Geometry, 9–17 July 1999, supported by the University of Warwick. We are grateful to fellow participant Martin Edjvet who encouraged us to consider the question. The first author was partially supported by the Australian Research Council and the third author was partially supported by an ESPRC grant.

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Received May 2000