SHORT BALANCED PRESENTATIONS OF PERFECT GROUPS

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Abstract

We report some initial results from an investigation of short balanced presentations of perfect groups. We determine the minimal length 2-generator balanced presentations for $SL_2(5)$ and $SL_2(7)$ and we show that $\widehat{M_{22}}$, the full covering group of the sporadic simple group M_{22}, has deficiency zero. We give presentations for $SL_2(7)$ and $\widehat{M_{22}}$ that are both new and of minimal length. We also determine the shortest 2-generator presentations for an infinite perfect group. This is done in the context of a complete classification of short 2-generator balanced presentations of perfect groups in terms of canonic presentations.

1 Introduction

Efficient presentations for groups have been the subject of much study. A survey of such presentations for simple groups and their full covering groups appears in [4], which provides relevant background material. Balanced presentations for perfect groups are efficient presentations which help us understand the answers to many associated questions.

Motivated by successful enumerations of balanced presentations of the trivial group [10, 13] we realized we could in an analogous way successfully enumerate short balanced presentations of general perfect groups. Here we report some initial results from a much more substantial enumeration which is underway. We completely classify all canonic 2-generator balanced presentations of perfect groups with relator length up to 17.

In particular we prove that the shortest 2-generator balanced presentations for the unique perfect group of order 120, $SL_2(5)$, have (relator) length 12. (This result was already implicit in previous enumerations [10, 13].) We also prove that the shortest 2-generator balanced presentations for the unique perfect group of order 336, $SL_2(7)$, have length 17. This result is new and improves greatly on the lengths of previously published balanced presentations for this group. We provide six canonic presentations of minimal length and use one to give a short efficient presentation of $PSL_2(7)$.

We give a balanced presentation for the covering group of the simple group M_{22}. This shows that this group, $\widehat{M_{22}}$, has deficiency zero, answering a previously unresolved question. Our canonic presentation is a unique representative of the shortest

[1]Both authors were partially supported by the Australian Research Council

possible presentations. Finally we give balanced presentations for an infinite perfect group.

The study of perfect groups is greatly facilitated by the classification provided by Holt and Plesken [11]. For example, it confirms that $SL_2(5)$ and $SL_2(7)$ are unique perfect groups with orders 120 and 336, respectively. Thus if coset enumeration reveals that a finitely presented group which is perfect has one of these orders then we know the group.

2 Technique

A presentation $\langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m \rangle$ is called *balanced* if $m = n$. We can enumerate all balanced presentations which are not too long and try to identify the groups. In fact we were motivated by an earlier enumeration where we studied potential counterexamples to the Andrews-Curtis conjecture. There we enumerated balanced presentations of the trivial group, the smallest perfect group. Here we are specifically not interested in the trivial group.

Combinatorial explosion means that not too long is pretty short in general. By restricting ourselves to perfect groups we can easily go further, far enough to be interesting. Our general selection criteria for presentations are as follows.

A) We insist that all generators are actually in the relators somewhere (ie, there is no obvious free quotient).

B) The group has trivial abelian quotient (ie, is perfect).

C) No relator (perhaps cycled and/or inverted) properly contains more than half of another. (Otherwise, there would exist a shorter "equivalent" presentation on the same generators.)

Note that these checks (and others later) are not independent; ie, B) implies A). However, this redundant checking is often helpful, since it can speed up the running time considerably. (Removing a presentation early means we do not need to run any further checks on it. Of course, this has to be balanced against the costs of the check.)

In this paper we only consider 2-generator presentations. However, similar principles apply for more generators. For two generators we add two more criteria.

D) We disallow relators of length 4 or less. Examination of the list of "canonic" relators, defined later, shows that these can either be reduced to one generator presentations (which are not interesting), or cannot satisfy B) above (ie, not perfect). Thus the smallest interesting presentation length is 10.

E) We disallow any relator which contains a generator precisely once, since these can be reduced to one generator cases (again, not interesting).

Throughout, we adopt the convention of using upper-case letters to denote inverses so that, for example, $A = a^{-1}$, etc. For two generators, we start by fixing on a set of generators $\{a, b\}$ and an ordering $a < A < b < B$, and a total presentation length ℓ. We then enumerate all balanced presentations on the generator set where the sum of the relator's lengths is ℓ, which satisfy our selection criteria. For example, to ensure that the abelian quotient is trivial we calculate the abelianised relators $(a^k b^m, a^n b^p)$ of each presentation, and discard those with $kp - mn \neq \pm 1$.

These criteria alone allow much redundancy. To partially address this we remove some equivalent presentations by putting each presentation into a standard form. By this we mean that: the relators are freely and cyclically reduced; each relator is the lexicographic minimum of all its cyclic permutations and their inverses; the relators are in order (length plus lexicographic).

For our purposes here, lists of presentations can also be pruned by retaining only a single representative of each orbit under the action of the automorphism group of the free group. For simplicity we initially only apply the length-preserving automorphisms of the extended symmetric group of order $2^n n!$ generated by the permutations $(x_i, x_j)(X_i, X_j)$, for all $i \neq j$, and (x_i, X_i), for all i. A *canonic* presentation is the least representative of such an orbit. For $n = 2$, this pruning decreases the list sizes by a factor of up to eight; for example, the list for $\ell = 10$ reduces to one. We do this to give us lists of presentations. What next?

The availability of packages for computational group theory, including GAP [8], MAGMA [1], Magnus [14] and `testisom` [12] makes it quite easy to experiment with groups. For coset enumeration we use the ACE enumerator [9] either as available in a package, or as a stand-alone program for some more difficult cases.

If a finitely presented group is finite then coset enumeration will reveal the fact (subject to space and time considerations and to general unsolvability results). So we simply try enumerating cosets of the trivial subgroup. There is a minor problem here: if we try to enumerate the cosets for an infinite group it could go on for a long time. In our initial study we allow a maximum of ten million cosets to be defined. With this limit, it takes less than one cpu minute for a coset enumeration to overflow on a reasonably fast machine.

3 Results

The following table indicates the results of the process. We see that most canonic balanced presentations for perfect groups in this length range actually define the trivial group. (This is another source of potential counterexamples to the Andrews-Curtis conjecture.)

Presentation length	No. of canonic presentations	Coset enumeration behaviour			
		trivial	120	336	overflow
10	1	1	-	-	-
11	4	4	-	-	-
12	7	6	1	-	-
13	68	65	3	-	-
14	78	77	-	-	1
15	600	590	5	-	5
16	694	688	6	-	-
17	6106	6057	38	6	5

Coset enumeration reveals that $SL_2(5)$ first appears at length 12. The unique canonic presentation is $\langle a, b \mid aabAb, abbbbaB \rangle$. This agrees with the results of the presentation enumerations in [10, 13]. As an immediate consequence we have:

Theorem 3.1 *The shortest 2-generator balanced presentations for $SL_2(5)$ have length 12.*

Much information on presentations of $SL_2(5)$ appears in [7]; it was the first perfect group known to have deficiency zero. A length 12 presentation already appears in [7, p. 69], in all editions except the first.

The situation is more complicated with $SL_2(7)$. It first obviously appears at length 17, six times in our table. To prove that these are indeed shortest requires further analysis of the shorter presentations which led to coset enumeration overflows. We have to discover enough about the groups defined by these presentations to show that they are not isomorphic to $SL_2(7)$. For our purposes this can easily be done by looking at low index subgroups.

The presentations of length 14 and 15 which led to overflows are:

$P_1 = \langle a, b \mid aabAb, abbbbbbaB\rangle$; $P_2 = \langle a, b \mid ababaB, aaaaaaabb\rangle$;
$P_3 = \langle a, b \mid abaBAB, aaaaaBAAB\rangle$; $P_4 = \langle a, b \mid abaBAB, aaaaBAbAB\rangle$;
$P_5 = \langle a, b \mid abaBAB, aaaBAbbAB\rangle$; $P_6 = \langle a, b \mid aabABAb, aaaBAbAB\rangle$.

We can readily prove that each of these present infinite groups by using programs like that in Example H20E40 [5, p. 306, vol. II]. Each presentation leads to one index 28 subgroup with infinite abelianization. In fact all six groups have 83 conjugacy classes of nontrivial subgroups up to index 28, suggesting that the groups are isomorphic. Indeed `testisom` quickly identifies isomorphisms between the groups. For example, subscripting generators according to presentation number, we have isomorphisms taking $a_1 \mapsto a_2b_2, b_1 \mapsto a_2$ and $a_2 \mapsto b_1, b_2 \mapsto B_1a_1$. These groups are all isomorphic to the full covering group of the triangle group $\langle 2, 3, 7\rangle$. Simply eliminating $z = B^2$ from $\langle a, b, z \mid b^2z, (ab)^3z, a^7Z\rangle$, which is a presentation for $\widehat{\langle 2, 3, 7\rangle}$, gives P_2.

Theorem 3.2 *The shortest 2-generator balanced presentations for $SL_2(7)$ have length 17.*

Theorem 3.3 *The shortest 2-generator balanced presentations for an infinite perfect group have length 14 and present $\widehat{\langle 2, 3, 7\rangle}$. The canonic presentation of length 14 for this group is $\langle a, b \mid aabAb, abbbbbbaB\rangle$.*

A previously published balanced presentation for $SL_2(7)$ [3] has length 37. Shorter presentations with length 27 are implicit from Conder's list [6] of one-relator quotients of the modular group combined with a result of Campbell, Havas, Hulpke and Robertson [2, Lemma 2.2]. Thus Conder's presentation 11.34 for $L_2(7)$:
$\langle x, y \mid x^3, y^2, xyxyxyxyXyXyxyXyxyXyXy\rangle$ leads to:
$\langle x, y \mid x^3y^2, xYxYxYxYXyXyxYXyxyXyXy\rangle$ as one presentation for $SL_2(7)$.

The six canonic presentations of length 17 revealed by our enumeration are:

$Q_1 = \langle a, b \mid abaBAB, aaaaBBaBaBB\rangle$; $Q_2 = \langle a, b \mid aaaabbb, ababaBAbAB\rangle$;
$Q_3 = \langle a, b \mid aabABAb, aaabbbAbAB\rangle$; $Q_4 = \langle a, b \mid aabABAb, aaaBABBBAB\rangle$;
$Q_5 = \langle a, b \mid aaabAbAb, abABBAbaB\rangle$; $Q_6 = \langle a, b \mid aabaaBAB, abbbaBBBB\rangle$.

It is easy to find the orders of the generators of each of the groups (eg, by doing coset enumerations over the subgroups generated by each of the generators). Subscripting generators according to presentation number we discover that:

$a_1^7, b_1^7; a_2^8, b_2^6; a_3^7, b_3^8; a_4^7, b_4^8; a_5^8, b_5^8; a_6^7, b_6^{14}$ are all trivial in the corresponding groups. It follows that all Q_i are on different generators except possibly Q_3 and Q_4. Since $\langle a, b \mid aabABAb, aaabbbAbAB, aaaBABBBAB\rangle$ also presents $SL_2(7)$ (readily revealed by coset enumeration), those presentations are on the same generators.

Further analysis reveals that a length 17 presentation for $SL_2(7)$ is implicit in Conder's list. Following Conder, define $u = xy$ and $v = Xy$. In terms of these generators his presentation 11.34 is: $\langle u, v \mid (vU)^3, (vUv)^2, uuuuvvuvuvv\rangle$. Multiplying the inverse of the first relator by the second gives the balanced presentation $\langle u, v \mid uVuvUv, uuuuvvuvuvv\rangle$, which is equivalent to our canonic presentation Q_1.

Presentation Q_2 has the obvious quotient $\langle a, b \mid a^4, b^3, ababaBAbAB\rangle$ which is a short presentation for $PSL_2(7)$, easily confirmed by coset enumeration.

Given that there were five coset enumeration overflows for length 17 canonic presentations, we need to further analyze those presentations to ensure that there are no other minimal length canonic presentations for $SL_2(7)$. The presentations which led to overflow are:

$R_1 = \langle a, b \mid abaBAB, aaaaaaBBABB\rangle$; $R_2 = \langle a, b \mid aabaBab, aaaaaabbAB\rangle$;
$R_3 = \langle a, b \mid aaabaBab, aaaBAbbAB\rangle$; $R_4 = \langle a, b \mid aaabAbAb, aabbbbbbb\rangle$;
$R_5 = \langle a, b \mid aababAAB, abbbbaBaB\rangle$.

Again we can use low index subgroups to investigate the associated groups. We discover that R_1, R_2, R_3 and R_4 also have one index 28 subgroup with infinite abelianization and have 83 conjugacy classes of nontrivial subgroups up to index 28. The groups defined by these R_i are also all isomorphic to $\widehat{\langle 2, 3, 7\rangle}$.

The group presented by R_5 is different. It has an index 22 subgroup, and the corresponding permutations generate M_{22}, the sporadic simple group with order 443520. This is enough to confirm that there are exactly six minimal length (17) canonic presentations for $SL_2(7)$.

However presentation R_5 merits further analysis. We do not easily find other low index subgroups. So we tried enumerating the cosets of $\langle a\rangle$ and $\langle b\rangle$. Both of those enumerations complete easily enough even restricted to a maximum of 10 million cosets. Encouraged by this we try the coset enumeration over the trivial subgroup again, allowing more cosets. Using the `Hard` strategy of ACE we discover the group has order 5322240 (defining a maximum of 20921635 cosets and total of 21611026). This is enough to tell us that R_5 presents $\widehat{M_{22}}$, the full covering group of M_{22}, and resolves in the positive the question about $\widehat{M_{22}}$ left open in [4].

Theorem 3.4 *The presentation $\langle a, b \mid aababAAB, abbbbaBaB\rangle$ defines $\widehat{M_{22}}$. It is the unique minimum length canonic presentation for the group.*

4 Conclusions

We have reported some initial results from an investigation of short balanced presentations of perfect groups. We have completely classified all canonic 2-generator balanced presentations of perfect groups with relator length up to 17. Given any presentation of such length it is straightforward to determine if it is interesting and, if so, to determine its canonic equivalent.

Our classification has revealed that only 5 isomorphism types arise up to this length: the trivial group; $SL_2(5)$ with order 120; $SL_2(7)$ with order 336; $\widehat{M_{22}}$ with order 5322240; and one infinite group, $\widehat{\langle 2,3,7 \rangle}$.

We are continuing the enumeration of canonic presentations of various kinds and expect many new results to be forthcoming. We anticipate finding new deficiency zero presentations for many perfect groups, including some for groups not currently known to have deficiency zero. We also expect to find other efficient presentations for various groups which are associated with perfect groups, in particular for quotients including simple quotients.

Acknowledgements

We are grateful to Marston Conder for helpful discussions.

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