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The $F^{a,b,c}$ conjecture is true, II ☆

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Abstract

In 1977 a five-part conjecture was made about a family of groups related to trivalent graphs and subsequently two parts of the conjecture were proved. The conjecture completely determines all finite members of the family. Here we complete the proof of the conjecture by giving proofs for the remaining three parts.
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1. Introduction

The groups $F^{a,b,c}$ are defined by

$$F^{a,b,c} = \langle r, s \mid r^2, rs^a rs^b rs^c \rangle.$$

They arose because some of the groups have Cayley diagrams which are '0-symmetric' [5] or 'faithful'. This notion was a consequence of work started in the 1920s by R.M. Foster

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on symmetrical graphs which could be used as electrical networks. At a conference held in Waterloo, Ontario, in April 1966, Foster presented a census of symmetric trivalent graphs with up to 400 vertices. (An extended version was published much later [6].) H.S.M. Coxeter, who had one of the few copies of Foster's original census, became interested. As part of Coxeter's investigation he considered the groups $F^{a,b,c}$.

Campbell, Coxeter and Robertson studied the groups in [1]. After determining the structure of various subclasses, they made 'the $F^{a,b,c}$ conjecture' which we state after some preliminaries. Combined with some results in [1] this conjecture completely describes the structure of all finite groups in the $F^{a,b,c}$ family in terms of a specific finite quotient which is fully understood.

Define $n = a + b + c$ and $d = (a - b, b - c)$. The structure of the groups

$$H^{a,b,c} = \langle r, s \mid r^2, s^{2n}, rs^a rs^b rs^c \rangle$$

is completely determined in [1, Section 3]. If $n = 0$ then $F^{a,b,c}$ is clearly infinite. In [1] the groups $F^{a,b,c}$ are shown to be infinite when $t = (a, b, c) \neq 1$ except when $H^{a/t,b/t,c/t}$ is abelian, in which case $F^{a,b,c} \cong H^{a,b,c} \cong C_{2n}$. We generalize this result in Section 3 for all $t \neq 1$ with $1 < d \leqslant 5$. The conjecture addresses the remaining cases. Provided $(a, b, c) = 1$, $n \neq 0$ and $(d, 6) \neq 6$, the groups $H^{a,b,c}$ are finite metabelian groups. If $(a, b, c) = 1$ and $d \geqslant 6$ the groups $F^{a,b,c}$ are infinite [1].

The $F^{a,b,c}$ conjecture is as follows. Suppose $(a, b, c) = 1$ and $n \neq 0$. Let $\theta : F^{a,b,c} \to H^{a,b,c}$ be the natural homomorphism. Let $N = \ker\theta$. Then

$$\begin{aligned}
&N = 1 \quad \text{if } d = 1,\\
&N = 1 \quad \text{if } d = 2,\\
&N \cong C_2 \quad \text{if } d = 3,\\
&N \cong Q_8 \quad \text{if } d = 4,\\
&N \cong SL(2, 5) \quad \text{if } d = 5.
\end{aligned}$$

The conjecture was proved true when $d = 1$ in [2], see also [3, Corollary 3.4] for an alternative proof. The conjecture was proved true when $d = 5$ in [9]. Many special cases supporting the conjecture have been proved, see [1,3,4,11,12].

In this paper we give the proof of the conjecture for the remaining three cases, $d = 2$, 3 and 4. These proofs were found by using computer-generated proofs [8] for specific instances which enabled us to observe the crucial role played by certain involutions. The paper [10] will explain how this was achieved.

As a consequence of these results we now assert that the theorem conjectured in [1] is true.

Theorem 1. *If $(a, b, c) = 1$ and $a + b + c \neq 0$ then the groups $F^{a,b,c}$ are finite if, and only if, $(a - b, b - c) \leqslant 5$. Indeed, the structure of these groups is based on the structure of their known quotients $H^{a,b,c}$ in a straightforward fashion.*

In the rest of this paper, we use the notation $x \sim y$ to mean that x commutes with y.

2. Preliminary results

In our investigation of the $F^{a,b,c}$ conjecture, we found it helpful to consider these groups written:

$$F^{a-jd,a,a+kd} = \langle r, s \mid r^2, rs^{a-jd}rs^a rs^{a+kd} \rangle,$$

for some $j, k \in \mathbb{Z}$ where $(j,k) = 1$. We use this notation throughout.

Before we give our proof of the conjecture, we set up some preliminary results and lemmas that form the building blocks of our proof.

For $F^{a-jd,a,a+kd}$ with $(j,k) = 1$, we have $n = 3a + kd - jd$ and $s^{2n} = s^{6a+2kd-2jd}$. In our proof we repeatedly use simple consequences of the defining relations. In particular, note that in $F^{a-jd,a,a+kd}$, we have

$$rs^{a-jd}r = s^{-a-kd}rs^{-a}, \tag{1}$$

$$rs^{a}r = s^{jd-a}rs^{-a-kd}, \tag{2}$$

and

$$rs^{a+kd}r = s^{-a}rs^{jd-a}. \tag{3}$$

These lead to the following less obvious relations which we use in our proofs:

$$rs^{2a-jd}r = \left(rs^{a-jd}r\right)\left(rs^{a}r\right) = s^{-a-kd}rs^{jd-2a}rs^{-a-kd}, \tag{4}$$

$$rs^{2a-jd}r = \left(rs^{a}r\right)\left(rs^{a-jd}r\right) = s^{jd-a}rs^{-2a-2kd}rs^{-a}, \tag{5}$$

$$rs^{2a+kd-jd}r = \left(rs^{a-jd}r\right)\left(rs^{a+kd}r\right) = s^{-a-kd}rs^{-2a}rs^{jd-a}, \tag{6}$$

$$rs^{2a+kd-jd}r = \left(rs^{a+kd}r\right)\left(rs^{a-jd}r\right) = s^{-a}rs^{-2a-kd+jd}rs^{-a}, \tag{7}$$

$$rs^{2a+kd}r = \left(rs^{a}r\right)\left(rs^{a+kd}r\right) = s^{jd-a}rs^{-2a-kd}rs^{jd-a}, \tag{8}$$

$$rs^{2a+kd}r = \left(rs^{a+kd}r\right)\left(rs^{a}r\right) = s^{-a}rs^{2jd-2a}rs^{-a-kd}, \tag{9}$$

$$rs^{jd}r = \left(rs^{a}r\right)\left(rs^{jd-a}r\right) = s^{jd-a}rs^{-kd}rs^{a+kd}, \tag{10}$$

$$rs^{jd}r = \left(rs^{jd-a}r\right)\left(rs^{a}r\right) = s^{a}rs^{kd+jd}rs^{-a-kd}, \tag{11}$$

$$rs^{kd}r = \left(rs^{-a}r\right)\left(rs^{a+kd}r\right) = s^{a+kd}rs^{-jd}rs^{jd-a}, \tag{12}$$

$$rs^{kd+jd}r = \left(rs^{jd-a}r\right)\left(rs^{a+kd}r\right) = s^{a}rs^{kd}rs^{jd-a}. \tag{13}$$

By using (4) and (5), (7) and (6), and (8) and (9), respectively, we obtain:

$$\begin{aligned} rs^{2a-jd}r &= s^{-a-kd}rs^{jd-2a}rs^{-a-kd} = s^{-a-kd}\left(s^{a}rs^{2a+2kd}rs^{a-jd}\right)s^{-a-kd} \\ &= s^{-kd}rs^{2a+2kd}rs^{-kd-jd}, \end{aligned} \tag{14}$$

$$rs^{2a+kd-jd}r = s^{-a}rs^{-2a-kd+jd}rs^{-a} = s^{-a}\left(s^{a-jd}rs^{2a}rs^{a+kd}\right)s^{-a}$$
$$= s^{-jd}rs^{2a}rs^{kd}, \tag{15}$$
$$rs^{2a+kd}r = s^{jd-a}rs^{-2a-kd}rs^{jd-a} = s^{jd-a}\left(s^{a+kd}rs^{2a-2jd}rs^{a}\right)s^{jd-a}$$
$$= s^{kd+jd}rs^{2a-2jd}rs^{jd}. \tag{16}$$

From these equations, we can prove the necessary lemmas.

Lemma 2. *In the groups $F^{a-jd,a,a+kd}$, with $j,k \in \mathbb{Z}$ and $(j,k)=1$,*

$$rs^{2n}r = rs^{6a+2kd-2jd}r \sim s^{d}.$$

Proof. First we show that $rs^{6a+2kd-2jd}r \sim s^{jd}$ and $rs^{6a+2kd-2jd}r \sim s^{kd}$, which implies the result since $(j,k)=1$. Thus, using Eqs. (14)–(16), we find

$$rs^{6a+2kd-2jd}r = \left(rs^{2a+kd-jd}r\right)\left(rs^{2a-jd}r\right)\left(rs^{2a+kd}r\right) = s^{-jd}rs^{6a+2kd-2jd}rs^{jd},$$
$$rs^{6a+2kd-2jd}r = \left(rs^{2a-jd}r\right)\left(rs^{2a+kd}r\right)\left(rs^{2a+kd-jd}r\right) = s^{-kd}rs^{6a+2kd-2jd}rs^{kd}.$$

Since $(j,k)=1$, $xj+yk=1$ for some integers x and y, and so

$$s^{-d}rs^{6a+2kd-2jd}rs^{d} = s^{-(xj+yk)d}rs^{6a+2kd-2jd}rs^{(xj+yk)d}$$
$$= s^{-xjd}.s^{-ykd}rs^{6a+2kd-2jd}rs^{ykd}.s^{xjd}$$
$$= s^{-xjd}rs^{6a+2kd-2jd}rs^{xjd} = rs^{6a+2kd-2jd}r. \qquad \square$$

We now prove that various families of elements have order (dividing) 2. The next five lemmas work towards a result, Lemma 8, which identifies a family of involutions that play a critical role in our proofs for each of the three remaining cases.

Lemma 3. *In $F^{a-jd,a,a+kd}$, for all integers $\ell \geqslant 0$,*

$$\left(rs^{2a+(\ell j)d+kd-jd}rs^{a-(\ell j)d}\right)^2 = 1 \tag{17}$$

and

$$\left(rs^{2a-(\ell j)d}rs^{a+(\ell j)d+kd-jd}\right)^2 = 1. \tag{18}$$

Proof. Our proof is by induction on ℓ, starting with two base cases. First, for $\ell = 0$

$$rs^{2a+kd-jd}rs^{a}rs^{2a+kd-jd}rs^{a}$$
$$= rs^{2a+kd-jd}(s^{jd-a}rs^{-a-kd})s^{2a+kd-jd}rs^{a} = rs^{a+kd}rs^{a-jd}rs^{a} = 1,$$
$$rs^{2a}rs^{a+kd-jd}rs^{2a}rs^{a+kd-jd}$$
$$= rs^{2a}rs^{a+kd-jd}\left(s^{jd}rs^{2a+kd-jd}rs^{-kd}\right)s^{a+kd-jd} \quad \text{(using (15))}$$

$$= rs^{2a}rs^{a+kd}rs^{2a+kd-jd}rs^{a-jd}$$
$$= rs^{2a}\left(s^{-a}rs^{jd-a}\right)s^{2a+kd-jd}rs^{a-jd} = rs^{a}rs^{a+kd}rs^{a-jd} = 1.$$

For $\ell = 1$

$$\begin{aligned} rs^{2a+kd}rs^{a-jd}rs^{2a+kd}rs^{a-jd} &= rs^{2a+kd}\left(s^{-a-kd}rs^{-a}\right)s^{2a+kd}rs^{a-jd} \\ &= rs^{a}rs^{a+kd}rs^{a-jd} = 1, \\ rs^{2a-jd}rs^{a+kd}rs^{2a-jd}rs^{a+kd} &= rs^{2a-jd}\left(s^{-a}rs^{jd-a}\right)s^{2a-jd}rs^{a+kd} \\ &= rs^{a-jd}rs^{a}rs^{a+kd} = 1. \end{aligned}$$

Now the inductive step: assume Lemma 3 is true for $0 \leqslant i < \ell$. Since we have shown it to be true for $\ell \in \{0, 1\}$, we can assume $\ell \geqslant 2$.

By the inductive hypothesis, since $0 \leqslant \ell - 2 < \ell$,

$$\left(rs^{2a+(\ell-2)jd+kd-jd}rs^{a-(\ell-2)jd}\right)^2 = \left(rs^{2a+\ell jd+kd-3jd}rs^{a-\ell jd+2jd}\right)^2 = 1, \quad (19)$$
$$\left(rs^{2a-(\ell-2)jd}rs^{a+(\ell-2)jd+kd-jd}\right)^2 = \left(rs^{2a-\ell jd+2jd}rs^{a+\ell jd+kd-3jd}\right)^2 = 1. \quad (20)$$

Also, since $0 < \ell - 1 < \ell$

$$\left(rs^{2a+(\ell-1)jd+kd-jd}rs^{a-(\ell-1)jd}\right)^2 = \left(rs^{2a+\ell jd+kd-2jd}rs^{a-\ell jd+jd}\right)^2 = 1, \quad (21)$$
$$\left(rs^{2a-(\ell-1)jd}rs^{a+(\ell-1)jd+kd-jd}\right)^2 = \left(rs^{2a-\ell jd+jd}rs^{a+\ell jd+kd-2jd}\right)^2 = 1. \quad (22)$$

We need some additional equations for our proof, namely

$$\left(rs^{2a-((\ell-1)j-k)d}rs^{a+((\ell-1)j-k)d+kd-jd}\right)^2 = \left(rs^{2a-\ell jd+kd+jd}rs^{a+\ell jd-2jd}\right)^2 = 1, \quad (23)$$
$$\left(rs^{2a+((\ell-1)j-k)d+kd-jd}rs^{a-((\ell-1)j-k)d}\right)^2 = \left(rs^{2a+\ell jd-2jd}rs^{a-\ell jd+kd+jd}\right)^2 = 1. \quad (24)$$

We prove (23) by using (10), (22), (19), (21), (1) and (2) to obtain

$$\begin{aligned} &rs^{2a-\ell jd+kd+jd}rs^{a+\ell jd-2jd}r \\ &\quad = rs^{2a-\ell jd+kd+jd}rs^{a+\ell jd-2jd}.r \\ &\quad \overset{(10)}{=} rs^{2a-\ell jd+kd+jd}.rs^{a+\ell jd+kd-2jd}r.s^{a-jd}rs^{jd}rs^{-a-kd} \\ &\quad \overset{(22)}{=} rs^{kd}rs^{-a-\ell jd-kd+2jd}.rs^{-a+\ell jd-2jd}r.s^{jd}rs^{-a-kd} \\ &\quad \overset{(19)}{=} rs^{kd}rs^{a-jd}rs^{a-\ell jd+2jd}.rs^{2a+\ell jd+kd-2jd}r.s^{-a-kd} \\ &\quad \overset{(21)}{=} rs^{kd}rs^{a-jd}rs^{jd}rs^{-2a-\ell jd-kd+2jd}rs^{-2a+\ell jd-kd-jd} \\ &\quad \overset{(1)}{=} rs^{-a}rs^{jd-a}rs^{-2a-\ell jd-kd+2jd}rs^{-2a+\ell jd-kd-jd} \\ &\quad \overset{(2)}{=} s^{-a-\ell jd+2jd}rs^{-2a+\ell jd-kd-jd}. \end{aligned}$$

Now, we prove (24) by using (3), (1), (21) and (23) to obtain

$$\begin{aligned}
&rs^{a-\ell jd+kd+jd}rs^{2a+\ell jd-2jd}r\\
&\quad= rs^{a-\ell jd+kd+jd}rs^{2a+\ell jd-2jd}.r\\
&\quad\overset{(3)}{=} rs^{a-\ell jd+kd+jd}.r.s^{3a+\ell jd+kd-2jd}rs^{a-jd}rs^{a}\\
&\quad\overset{(1)}{=} rs^{-\ell jd+kd+2jd}rs^{-a-kd}.rs^{2a+\ell jd+kd-2jd}r.s^{a-jd}rs^{a}\\
&\quad\overset{(21)}{=} rs^{-\ell jd+kd+2jd}.rs^{-2a+\ell jd-kd-jd}r.s^{-2a-\ell jd-kd+2jd}rs^{\ell jd-2jd}rs^{a}\\
&\quad\overset{(23)}{=} rs^{a+kd}rs^{2a-\ell jd+kd+jd}.rs^{-a-kd}r.s^{\ell jd-2jd}rs^{a}\\
&\quad\overset{(1)}{=} .rs^{a+kd}r.s^{3a-\ell jd+kd}rs^{a+\ell jd-2jd}rs^{a}\\
&\quad\overset{(3)}{=} s^{-a}.rs^{2a-\ell jd+kd+jd}rs^{a+\ell jd-2jd}r.s^{a}\\
&\quad\overset{(23)}{=} s^{-2a-\ell jd+2jd}rs^{-a+\ell jd-kd-jd}.
\end{aligned}$$

Next, we complete the inductive proof for the second part (18) of the lemma by using (23), (13), (24), (2), (21) and (3) to obtain

$$\begin{aligned}
&rs^{2a-\ell jd}rs^{a+\ell jd+kd-jd}r\\
&\quad= rs^{2a-\ell jd}\left(rs^{a+\ell jd-2jd}r\right)\left(rs^{kd+jd}r\right)\\
&\quad\overset{(23),\,(13)}{=} rs^{-kd-jd}rs^{-a-\ell jd+2jd}.rs^{-a+\ell jd-kd-jd}r.s^{kd}rs^{jd-a}\\
&\quad\overset{(24)}{=} rs^{-kd-jd}rs^{a}rs^{a-\ell jd+kd+jd}rs^{2a+\ell jd+kd-2jd}rs^{jd-a}\\
&\quad\overset{(2)}{=} rs^{-kd-jd}\left(s^{jd-a}rs^{-a-kd}\right)s^{a-\ell jd+kd+jd}rs^{2a+\ell jd+kd-2jd}rs^{jd-a}\\
&\quad= rs^{-a-kd}rs^{-\ell jd+jd}.rs^{2a+\ell jd+kd-2jd}r.s^{jd-a}\\
&\quad\overset{(21)}{=} rs^{-a-kd}rs^{-a}rs^{-2a-\ell jd-kd+2jd}rs^{-2a+\ell jd}\\
&\quad\overset{(3)}{=} s^{-a-\ell jd-kd+jd}rs^{-2a+\ell jd}.
\end{aligned}$$

Finally, we complete the inductive proof for the first part (17) of the lemma by using (21), (11), (23), (3), the (now proved) second part of this lemma and (1) to obtain

$$\begin{aligned}
&rs^{2a+\ell jd+kd-jd}rs^{a-\ell jd}r\\
&\quad= rs^{2a+\ell jd+kd-jd}\left(rs^{a-\ell jd+jd}r\right)\left(rs^{-jd}r\right)\\
&\quad\overset{(21),\,(11)}{=} rs^{jd}rs^{-a+\ell jd-jd}rs^{-a-\ell jd+2jd}rs^{-kd-jd}rs^{-a}\\
&\quad= rs^{jd}rs^{-a+\ell jd-jd}.rs^{-a-\ell jd+2jd}r.s^{-kd-jd}rs^{-a}\\
&\quad\overset{(23)}{=} rs^{jd}rs^{a+kd}rs^{a+\ell jd-2jd}rs^{2a-\ell jd}rs^{-a}\\
&\quad\overset{(3)}{=} rs^{jd}\left(s^{-a}rs^{jd-a}\right)s^{a+\ell jd-2jd}rs^{2a-\ell jd}rs^{-a}
\end{aligned}$$

$$
\begin{aligned}
&= rs^{jd-a}rs^{\ell jd-jd}.rs^{2a-\ell jd}r.s^{-a}\\
&\overset{(18)}{=} rs^{jd-a}rs^{-a-kd}rs^{-2a+\ell jd}rs^{-2a-\ell jd-kd+jd}\\
&\overset{(1)}{=} s^{-a+\ell jd}rs^{-2a-\ell jd-kd+jd}. \qquad \square
\end{aligned}
$$

The same kind of arguments allow us to extend Lemma 3 to negative ℓ. Thus we have

Lemma 4. *In $F^{a-jd,a,a+kd}$, for all integers ℓ,*

$$\left(rs^{2a+(\ell j)d+kd-jd}rs^{a-(\ell j)d}\right)^2 = 1$$

and

$$\left(rs^{2a-(\ell j)d}rs^{a+(\ell j)d+kd-jd}\right)^2 = 1.$$

Proof. Let us consider an integer $\ell < 0$. Using $i = -\ell$ for an integer $i > 0$, we have

$$
\begin{aligned}
\left(rs^{2a+(\ell j)d+kd-jd}rs^{a-(\ell j)d}\right)^2 &= \left(rs^{2a-ijd+kd-jd}rs^{a+ijd}\right)^2\\
&= \left(rs^{2a-((i+1)j-k)d}rs^{a+((i+1)j-k)d+kd-jd}\right)^2
\end{aligned}
$$

and

$$
\begin{aligned}
\left(rs^{2a-(\ell j)d}rs^{a+(\ell j)d+kd-jd}\right)^2 &= \left(rs^{2a+ijd}rs^{a-ijd+kd-jd}\right)^2\\
&= \left(rs^{2a+((i+1)j-k)d+kd-jd}rs^{a-((i+1)j-k)d}\right)^2.
\end{aligned}
$$

Thus, a proof of $(i+1)j-k$ is equivalent to one of ℓj. Obviously, as $i > 0$, then $i+2 > 0$ and so, the two expressions equal the identity for $\ell = i+2$ according to our induction proof. During the proof of ℓj, however, we found that the two expressions were also trivial for $(\ell-1)j-k$. Thus for $\ell = i+2$, we have the equations true for $(i+2-1)j-k = (i+1)j-k$ as required, and so, the proof holds for all negative integers as well. $\square$

We now extend the lemma further, again by induction.

Lemma 5. *In $F^{a-jd,a,a+kd}$, for all integers ℓ and $m \geqslant 0$,*

$$\left(rs^{2a+(\ell j-mk)d+kd-jd}rs^{a-(\ell j-mk)d}\right)^2 = 1 \tag{25}$$

and

$$\left(rs^{2a-(\ell j-mk)d}rs^{a+(\ell j-mk)d+kd-jd}\right)^2 = 1. \tag{26}$$

Proof. Our proof is by induction on m, starting with two base cases. First, for $m = 0$ this is Lemma 4 and for $m = 1$ it is obtained by putting $\ell + 1$ for ℓ in Lemma 4.

Now the inductive step: assume Lemma 5 is true for $0 \leqslant i < m$. Since we have shown it to be true for $m \in \{0, 1\}$, we can assume $m \geqslant 2$.

By the inductive hypothesis, since $0 < m - 1 < m$

$$\begin{aligned}
&\left(rs^{2a+(\ell j-(m-1)k)d+kd-jd}rs^{a-(\ell j-(m-1)k)d}\right)^2 \\
&\quad = \left(rs^{2a+\ell jd-mkd+2kd-jd}rs^{a-\ell jd+mkd-kd}\right)^2 = 1, \qquad (27)\\
&\left(rs^{2a-(\ell j-(m-1)k)d}rs^{a+(\ell j-(m-1)k)d+kd-jd}\right)^2 \\
&\quad = \left(rs^{2a-\ell jd+mkd-kd}rs^{a+\ell jd-mkd+2kd-jd}\right)^2 = 1. \qquad (28)
\end{aligned}$$

Substituting $\ell - 1$ and $\ell + 1$ for ℓ in these we deduce

$$\begin{aligned}
&\left(rs^{2a+(\ell-1)jd-mkd+2kd-jd}rs^{a-(\ell-1)jd+mkd-kd}\right)^2 \\
&\quad = \left(rs^{2a+\ell jd-mkd+2kd-2jd}rs^{a-\ell jd+mkd-kd+jd}\right)^2 = 1, \qquad (29)\\
&\left(rs^{2a-(\ell-1)jd+mkd-kd}rs^{a+(\ell-1)jd-mkd+2kd-jd}\right)^2 \\
&\quad = \left(rs^{2a-\ell jd+mkd-kd+jd}rs^{a+\ell jd-mkd+2kd-2jd}\right)^2 = 1, \qquad (30)\\
&\left(rs^{2a+(\ell+1)jd-mkd+2kd-jd}rs^{a-(\ell+1)jd+mkd-kd}\right)^2 \\
&\quad = \left(rs^{2a+\ell jd-mkd+2kd}rs^{a-\ell jd+mkd-kd-jd}\right)^2 = 1. \qquad (31)
\end{aligned}$$

Next, we complete the inductive proof for the second part (26) of the lemma by using (12), (28), (29) and (27) to obtain

$$\begin{aligned}
&rs^{2a-(\ell j-mk)d}rs^{a+(\ell j-mk)d+kd-jd}r \\
&\quad = rs^{2a-\ell jd+mkd}rs^{a+\ell jd-mkd+kd-jd}.r \\
&\quad \overset{(12)}{=} rs^{2a-\ell jd+mkd}rs^{a+\ell jd-mkd+2kd-jd}rs^{a-jd}rs^{jd}rs^{-a-kd} \\
&\quad = rs^{2a-\ell jd+mkd}.rs^{a+\ell jd-mkd+2kd-jd}r.s^{a-jd}rs^{jd}rs^{-a-kd} \\
&\quad \overset{(28)}{=} rs^{kd}rs^{-a-\ell jd+mkd-2kd+jd}.rs^{-a+\ell jd-mkd+kd-jd}r.s^{jd}rs^{-a-kd} \\
&\quad \overset{(29)}{=} rs^{kd}rs^{a-jd}rs^{a-\ell jd+mkd-kd+jd}.rs^{2a+\ell jd-mkd+2kd-jd}r.s^{-a-kd} \\
&\quad \overset{(27)}{=} rs^{kd}rs^{a-jd}rs^{jd}rs^{-2a-\ell jd+mkd-2kd+jd}rs^{-2a+\ell jd-mkd} \\
&\quad \overset{(12)}{=} s^{-a-\ell jd+mkd-kd+jd}rs^{-2a+\ell jd-mkd} \\
&\quad = s^{-a-(\ell j-mk)d-kd+jd}rs^{-2a+(\ell j-mk)d}.
\end{aligned}$$

Substituting $\ell + 1$ for ℓ in this we deduce

$$\begin{aligned}&\left(rs^{2a-((\ell+1)j-mk)d}rs^{a+((\ell+1)j-mk)d+kd-jd}\right)^2\\&\quad=\left(rs^{2a-\ell jd+mkd-jd}rs^{a+\ell jd-mkd+kd}\right)^2=1.\end{aligned}\tag{32}$$

Finally, we complete the inductive proof for the first part (26) of the lemma using (11), (31), (32) and the (now proved) second part of this lemma to obtain

$$\begin{aligned}&rs^{a-(\ell j-mk)d}rs^{2a+(\ell j-mk)d+kd-jd}r\\&\quad=rs^{a-\ell jd+mkd}rs^{2a+\ell jd-mkd+kd-jd}.r\\&\quad\overset{(11)}{=}rs^{a-\ell jd+mkd}.rs^{2a+\ell jd-mkd+2kd}r.s^{-a-kd}rs^{-jd}rs^{a}\\&\quad\overset{(31)}{=}rs^{kd+jd}rs^{-2a-\ell jd+mkd-2kd}.rs^{-2a+\ell jd-mkd+jd}r.s^{-jd}rs^{a}\\&\quad\overset{(32)}{=}rs^{kd+jd}rs^{-a-kd}rs^{2a-\ell jd+mkd-jd}.rs^{a+\ell jd-mkd+kd-jd}r.s^{a}\\&\quad\overset{(26)}{=}rs^{kd+jd}rs^{-a-kd}rs^{-jd}rs^{-a-\ell jd+mkd-kd+jd}rs^{-a+\ell jd-mkd}\\&\quad\overset{(11)}{=}s^{-2a-\ell jd+mkd-kd+jd}rs^{-a+\ell jd-mkd}\\&\quad=s^{-2a-(\ell j-mk)d-kd+jd}rs^{-a+(\ell j-mk)d}.\qquad\square\end{aligned}$$

We now show that Lemma 5 also holds for all $m\leqslant 0$. To do this we prove the lemma with $-m$ in place of m.

Lemma 6. *In $F^{a-jd,a,a+kd}$, for all integers ℓ and $m\geqslant 0$,*

$$\left(rs^{2a+(\ell j+mk)d+kd-jd}rs^{a-(\ell j+mk)d}\right)^2=1\tag{33}$$

and

$$\left(rs^{2a-(\ell j+mk)d}rs^{a+(\ell j+mk)d+kd-jd}\right)^2=1.\tag{34}$$

Proof. Our proof is by induction on m, starting with one base case. For $m=0$ this is Lemma 4.

Now the inductive step: assume Lemma 6 is true for $0\leqslant i<m$. Since we have shown it to be true for $m=0$, we can assume $m\geqslant 1$. By the inductive hypothesis, since $0\leqslant m-1<m$

$$\begin{aligned}&\left(rs^{2a+(\ell j+(m-1)k)d+kd-jd}rs^{a-(\ell j+(m-1)k)d}\right)^2\\&\quad=\left(rs^{2a+\ell jd+mkd-jd}rs^{a-\ell jd-mkd+kd}\right)^2=1,\end{aligned}\tag{35}$$

$$\begin{aligned}&\left(rs^{2a-(\ell j+(m-1)k)d}rs^{a+(\ell j+(m-1)k)d+kd-jd}\right)^2\\&\quad=\left(rs^{2a-\ell jd-mkd+kd}rs^{a+\ell jd+mkd-jd}\right)^2=1.\end{aligned}\tag{36}$$

Substituting $\ell-1$ for ℓ in these we deduce

$$\begin{aligned}&\left(rs^{2a+(\ell-1)jd+mkd-jd}rs^{a-(\ell-1)jd-mkd+kd}\right)^2\\&\quad=\left(rs^{2a+\ell jd+mkd-2jd}rs^{a-\ell jd-mkd+kd+jd}\right)^2=1,\end{aligned}\tag{37}$$

$$\begin{aligned}&\left(rs^{2a-(\ell-1)jd-mkd+kd}rs^{a+(\ell-1)jd+mkd-jd}\right)^2\\&\quad=\left(rs^{2a-\ell jd-mkd+kd+jd}rs^{a+\ell jd+mkd-2jd}\right)^2=1.\end{aligned}\tag{38}$$

Using (11), (37), (38), (36), (2) and (1) we obtain

$$\begin{aligned}&rs^{a-(\ell j+mk)d}rs^{2a+(\ell j+mk)d+kd-jd}r\\&\quad=rs^{a-\ell jd-mkd}rs^{2a+\ell jd+mkd+kd-jd}.r\\&\quad\overset{(11)}{=}rs^{a-\ell jd-mkd}.rs^{2a+\ell jd+mkd-2jd}r.s^{-a}rs^{jd}rs^{a+kd}\\&\quad\overset{(37)}{=}rs^{-kd-jd}rs^{-2a-\ell jd-mkd+2jd}.rs^{-2a+\ell jd+mkd-kd-jd}r.s^{jd}rs^{a+kd}\\&\quad\overset{(38)}{=}rs^{-kd-jd}rs^{-a}rs^{2a-\ell jd-mkd+kd+jd}.rs^{a+\ell jd+mkd-jd}r.s^{a+kd}\\&\quad\overset{(36)}{=}rs^{-kd-jd}rs^{-a}rs^{jd}rs^{-a-\ell jd-mkd+jd}rs^{-a+\ell jd+mkd}\\&\quad\overset{(2)}{=}rs^{a-jd}rs^{a}rs^{-a-\ell jd-mkd+jd}rs^{-a+\ell jd+mkd}\\&\quad\overset{(1)}{=}s^{-2a-\ell jd-mkd-kd+jd}rs^{-a+\ell jd+mkd},\end{aligned}$$

which completes the inductive proof for the first part (33) of the lemma. Substituting $\ell-1$ for ℓ in this we deduce

$$\begin{aligned}&\left(rs^{2a+((\ell-1)j+mk)d+kd-jd}rs^{a-((\ell-1)j+mk)d}\right)^2\\&\quad=\left(rs^{2a+\ell jd+mkd+kd-2jd}rs^{a-\ell jd-mkd+jd}\right)^2=1.\end{aligned}\tag{39}$$

Finally, we complete the inductive proof for the second part (34) of the lemma by using (10), (36), the (now proved) first part of this lemma, (39), (3) and (1) to obtain

$$\begin{aligned}&rs^{2a-(\ell j+mk)d}rs^{a+(\ell j+mk)d+kd-jd}r\\&\quad=rs^{2a-\ell jd-mkd}rs^{a+\ell jd+mkd+kd-jd}.r\\&\quad\overset{(10)}{=}rs^{2a-\ell jd-mkd}.rs^{a+\ell jd+mkd-jd}r.s^{a+kd}rs^{-jd}rs^{jd-a}\\&\quad\overset{(36)}{=}rs^{-kd}rs^{-a-\ell jd-mkd+jd}.rs^{-a+\ell jd+mkd}r.s^{-jd}rs^{jd-a}\\&\quad\overset{(33)}{=}rs^{-kd}rs^{a+kd}rs^{a-\ell jd-mkd}.rs^{2a+\ell jd+mkd+kd-2jd}r.s^{jd-a}\\&\quad\overset{(39)}{=}rs^{-kd}rs^{a+kd}rs^{-jd}rs^{-2a-\ell jd-mkd-kd+2jd}rs^{-2a+\ell jd+mkd}\\&\quad\overset{(3)}{=}rs^{-a-kd}rs^{-a}rs^{-2a-\ell jd-mkd-kd+2jd}rs^{-2a+\ell jd+mkd}\\&\quad\overset{(1)}{=}s^{-a-\ell jd-mkd-kd+jd}rs^{-2a+\ell jd+mkd}.\qquad\square\end{aligned}$$

By combining the previous two lemmas, we obtain

Lemma 7. *In $F^{a-jd,a,a+kd}$, for all integers ℓ and m,*

$$\left(rs^{2a+(\ell j+mk)d+kd-jd}rs^{a-(\ell j+mk)d}\right)^2 = 1$$

and

$$\left(rs^{2a-(\ell j+mk)d}rs^{a+(\ell j+mk)d+kd-jd}\right)^2 = 1.$$

We now present a key result which generalizes and encompasses the previous five lemmas.

Lemma 8. *In $F^{a-jd,a,a+kd}$, for all integers i,*

$$\left(rs^{2a+id+kd-jd}rs^{a-id}\right)^2 = 1$$

and

$$\left(rs^{2a-id}rs^{a+id+kd-jd}\right)^2 = 1.$$

Proof.

Since $(j,k) = 1$, there exist integers x and y such that $xj + yk = 1$. Thus, for any integer i, we have $ixj + iyk = i$. Setting $\ell = ix$ and $m = iy$ and using Lemma 7 we have

$$\begin{aligned}\left(rs^{2a+id+kd-jd}rs^{a-id}\right)^2 &= \left(rs^{2a+ixjd+iykd+kd-jd}rs^{a-ixjd-iykd}\right)^2\\ &= \left(rs^{2a+(ixj+iyk)d+kd-jd}rs^{a-(ixj+iyk)d}\right)^2\\ &= \left(rs^{2a+(\ell j+mk)d+kd-jd}rs^{a-(\ell j+mk)d}\right)^2 = 1\end{aligned}$$

and

$$\begin{aligned}\left(rs^{2a-id}rs^{a+id+kd-jd}\right)^2 &= \left(rs^{2a-ixjd-iykd}rs^{a+ixjd+iykd+kd-jd}\right)^2\\ &= \left(rs^{2a-(ixj+iyk)d}rs^{a+(ixj+iyk)d+kd-jd}\right)^2\\ &= \left(rs^{2a-(\ell j+mk)d}rs^{a+(\ell j+mk)d+kd-jd}\right)^2 = 1.\end{aligned}$$

□

We also require some results from former work, which we state, without proof.

Lemma 9 *(*(Campbell, Coxeter and Robertson [1, Lemma 2.1])).

$$\begin{aligned}F^{a,b,c} &= F^{b,c,a} = F^{c,a,b} = F^{-c,-b,-a} = F^{-b,-a,-c} = F^{-a,-c,-b}\\ &\cong F^{a,c,b} = F^{c,b,a} = F^{b,a,c} = F^{-b,-c,-a} = F^{-c,-a,-b} = F^{-a,-b,-c}.\end{aligned}$$

This implies that we need only consider cases where $a + b + c \geqslant 0$ and $a \leqslant b \leqslant c$.

3. $F^{a,b,c}$ when $(a,b,c) \neq 1$

The $F^{a,b,c}$ conjecture handles the situation when $(a,b,c) = 1$. Before resolving the conjecture we provide a new result about the situation when a, b and c have a nontrivial common divisor. As we noted in Section 1, it was shown in [1] that when $(a,b,c) \neq 1$, then $F^{a,b,c} \cong H^{a,b,c}$ in the case that $H^{a/t,b/t,c/t}$ is abelian, where $t = (a,b,c)$. We prove in this section that, rather surprisingly, $F^{a,b,c} \cong H^{a,b,c}$ whenever $(a,b,c) \neq 1$ and $1 < d \leqslant 5$. This generalizes the earlier result and extends it to some infinite groups.

Theorem 10. *Let* $F^{a,b,c} = \langle r, s \mid r^2, rs^a rs^b rs^c \rangle$. *Define* $n = a+b+c$ *and* $d = (a-b, b-c)$ *and let* $H^{a,b,c} = \langle r, s \mid r^2, s^{2n}, rs^a rs^b rs^c \rangle$. *If* $(a,b,c) \neq 1$ *with* $n = a+b+c \neq 0$, *then for* $1 < d \leqslant 5$,

$$F^{a,b,c} \cong H^{a,b,c}.$$

Proof. Using the notation $F^{a-jd,a,a+kd}$, with $(j,k) = 1$, we have $s^{2n} = s^{6a+2kd-2jd}$. Consider $d \in \{1,2,3,4,5\}$, where $(a-jd, a, a+kd) = t \neq 1$. Obviously, a is a multiple of t, and so t divides both kd and jd. Given $(j,k) = 1$, then $(t,d) \neq 1$, for if this were not the case, then $t \neq 1$ would be a common factor of j and k.

Let $(t,d) = u \neq 1$. Here $d/u \in \mathbb{Z}$, and $t/u \in \mathbb{Z}$, with $(t/u, d/u) = 1$. As t divides both jd and kd, t/u divides both jd/u and kd/u. However, $(t/u, d/u) = 1$, so t/u is a common factor of j and k. This implies that $t = u$, and t divides d. Obviously where $d = 1$, we cannot have $t \neq 1$ which divides d. So, for $d \in \{2,3,4,5\}$ with $t \neq 1$ then $(a,d) \neq 1$ we have one of two cases: a is a multiple of d; or $d = 4$ and a is a multiple of 2 but not of 4.

If a is a multiple of d, then all of $a-jd$, a, and $a+kd$ are divisible by d and as such, are not coprime. If we consider $F^{a-jd,a,a+kd}$ where $a = md$ for some integer m, then using $i = m$ in the first part of Lemma 8 and simplifying gives $s^{6a+2kd-2jd} = 1$, as required.

Now consider $d = 4$ and $a = 2m$ for some odd integer m. Using $i = m$ in the second part of Lemma 8 and simplifying gives $s^{6a+8k-8j} = 1$, as required. □

4. Proof of the $F^{a,b,c}$ conjecture

The conjecture is given in Section 1. For the group $F^{a,b,c} = \langle r, s \mid r^2, rs^a rs^b rs^c \rangle$, we continue using the equivalent $F^{a-jd,a,a+kd}$ for $j,k \in \mathbb{Z}$ with $(j,k) = 1$, and drop superscripts where convenient. We define $n = a+b+c$ and $d = (a-b, b-c)$ and we suppose that $(a,b,c) = 1$, $n \neq 0$ and $(d,6) \neq 6$.

Using Lemma 9, we know that for $F^{a-jd,a,a+kd}$, we can assume, without loss of generality, that $a - jd \leqslant a \leqslant a + kd$ and thus, d, j, and k must all have the same sign, and further, without loss of generality, that they are positive. We prove each of the three remaining cases ($d = 2$, $d = 3$ and $d = 4$) of the conjecture in turn. We note that the same kind of proof as used for $d = 2$ works for $d = 1$, however the proof for $d = 5$ (see [9]) is quite different.

4.1. $d = 2$

For $d = 2$, the groups have the form $F^{a-2j,a,a+2k}$, and $n = 3a + 2k - 2j$. It suffices to show that $s^{2n} = s^{6a+4k-4j} = 1$ in $F^{a-2j,a,a+2k}$. Using $i = a$ in the second part of Lemma 8 and simplifying gives the result easily. (We note that a similar proof, this time using the first part of Lemma 8, works for $d = 1$, providing an alternative to the two published proofs [2,3].)

4.2. $d = 3$

For $d = 3$, the groups have the form $F^{a-3j,a,a+3k}$, and $n = 3a + 3k - 3j$. We need to show that $N = \ker\theta : F \to H \cong C_2$. Thus we need to show that s^{2n} has order 2 and is central in F.

First we prove that s^{2n} is central in F by putting $i = a$ in the second part of Lemma 8 and using the relator $rs^{a-3j}rs^{a}rs^{a+3k}$ twice to obtain

$$\begin{aligned} 1 &= s^{4a+3k-3j}rs^{-a}rs^{4a+3k-3j}rs^{-a}r \\ &= s^{4a+3k-3j}\left(s^{a+3k}rs^{a-3j}\right)s^{4a+3k-3j}\left(s^{a+3k}rs^{a-3j}\right) \\ &= s^{6a+6k-6j}rs^{6a+6k-6j}r. \end{aligned}$$

Now, using the relator $rs^{a-3j}rs^{a}rs^{a+3k}$ and this result, we deduce

$$\begin{aligned} s^{-6a-6k+6j} &= \left(rs^{a-3j}rs^{a}rs^{a+3k}\right)s^{-6a-6k+6j} = rs^{a-3j}rs^{a}.rs^{-6a-6k+6j}.s^{a+3k} \\ &= rs^{a-3j}rs^{7a+6k-6j}rs^{a+3k} = rs^{a-3j}.rs^{6a+6k-6j}.s^{a}rs^{a+3k} \\ &= rs^{-5a-6k+3j}rs^{a}rs^{a+3k} = rs^{-6a-6k+6j}.s^{a-3j}rs^{a}rs^{a+3k} \\ &= s^{6a+6k-6j}rs^{a-3j}rs^{a}rs^{a+3k} = s^{6a+6k-6j}. \end{aligned}$$

Thus s^{2n} has order dividing 2. It remains to show that s^{2n} is nontrivial. This is true because [3, Theorem 3.3] shows that in this case H has multiplier C_2.

4.3. $d = 4$

The case $d = 4$ requires more work than the previous cases as the kernel is more complicated. The groups have the form $F^{a-4j,a,a+4k}$, and $n = 3a + 4k - 4j$. We need to show that $N = \ker\theta : F \to H \cong Q_8$. It suffices to show that s^{2n} has order 4, s^{4n} is central in F, and that $N = \langle s^{2n}, rs^{2n}r\rangle$.

Here Lemma 2 shows that $rs^{6a+8k-8j}r \sim s^4$. Setting $i = a$ and $i = 0$ in the second and first parts of Lemma 8, respectively, we deduce

$$rs^{-2a}rs^{5a+4k-4j}rs^{-2a}rs^{5a+4k-4j} = 1 \quad \text{and} \quad rs^{2a+4k-4j}rs^{a}rs^{2a+4k-4j}rs^{a} = 1.$$

So, as $rs^{2a}r = s^{4j}rs^{2a+4k-4j}rs^{-4k}$ from (15), we have

$$
\begin{aligned}
1 &= rs^{-2a}rs^{5a+4k-4j}rs^{-2a}rs^{5a+4k-4j}\\
&= \left(s^{4k}rs^{-2a-4k+4j}rs^{-4j}\right)s^{5a+4k-4j}\left(s^{4k}rs^{-2a-4k+4j}rs^{-4j}\right)s^{5a+4k-4j}\\
&= s^{4k}rs^{-2a-4k+4j}rs^{5a+8k-8j}rs^{-2a-4k+4j}rs^{5a+4k-8j}\\
&= rs^{-2a-4k+4j}rs^{5a+8k-8j}rs^{-2a-4k+4j}rs^{5a+8k-8j}\\
&= rs^{-2a-4k+4j}rs^{5a+8k-8j}\left(s^{a}rs^{2a+4k-4j}rs^{a}\right)s^{5a+8k-8j}\\
&= rs^{-2a-4k+4j}rs^{6a+8k-8j}rs^{2a+4k-4j}rs^{6a+8k-8j}.
\end{aligned}
$$

Since $(a-4j, a, a+4k) = 1$, a must be odd. It follows that $2a+4k-4j+2$ is divisible by 4 so, since $rs^{6a+8k-8j}r \sim s^4$ we deduce that $rs^{6a+8k-8j}r \sim s^{2a+4k-4j+2}$. Thus

$$
1 = rs^{-2a-4k+4j}rs^{6a+8k-8j}rs^{2a+4k-4j}rs^{6a+8k-8j} = rs^2rs^{6a+8k-8j}rs^{-2}rs^{6a+8k-8j},
$$

which tells us that $s^{-2}rs^{6a+8k-8j}rs^2 = rs^{-6a-8k+8j}r$.

Likewise, since a is odd, $6a+8k-8j \equiv 2 \bmod 4$. Therefore,

$$
\begin{aligned}
s^{-6a-8k+8j}rs^{6a+8k-8j}rs^{6a+8k-8j} &= rs^{-6a-8k+8j}r, \qquad (40)\\
s^{6a+8k-8j}rs^{6a+8k-8j}rs^{-6a-8k+8j} &= rs^{-6a-8k+8j}r,
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
s^{6a+8k-8j}rs^{6a+8k-8j} &= rs^{6a+8k-8j}rs^{-6a-8k+8j}r, \qquad (41)\\
s^{6a+8k-8j}rs^{6a+8k-8j} &= rs^{-6a-8k+8j}rs^{6a+8k-8j}r. \qquad (42)
\end{aligned}
$$

Hence, equating the right-hand sides of (41) and (42),

$$
\begin{aligned}
&rs^{6a+8k-8j}rs^{-6a-8k+8j}r\\
&\quad = rs^{-6a-8k+8j}rs^{6a+8k-8j}r \quad \Rightarrow \quad s^{12a+16k-16j} = rs^{12a+16k-16j}r, \qquad (43)
\end{aligned}
$$

which proves that s^{4n} is central.

Also,

$$
\begin{aligned}
&rs^{6a+8k-8j}rs^{-6a-8k+8j}r\\
&\quad = \left(rs^{6a+8k-8j}rs^{-6a-8k+8j}r\right)^{-1}\\
&\quad \overset{(41)}{\Rightarrow} \quad s^{6a+8k-8j}rs^{6a+8k-8j} = s^{-6a-8k+8j}rs^{-6a-8k+8j}\\
&\quad \Rightarrow \quad s^{12a+16k-16j} = rs^{-12a-16k+16j}r. \qquad (44)
\end{aligned}
$$

So, equating the right-hand sides of (43) and (44), we deduce that s^{2n} has order dividing 4.

Next we show that $N = \langle s^{2n}, rs^{2n}r\rangle$. Let $x = s^{6a+8k-8j}$ and $y = rs^{6a+8k-8j}r$ and consider $\langle x, y\rangle$, which is obviously a subgroup of N. It is easily seen that $rxr = y$, $ryr = x$ and $s^{-1}xs = x$, and so it remains to show that $s^{-1}ys \in \langle x, y\rangle$.

Setting $i = a + k - j$ in the first part of Lemma 8, we deduce

$$1 = rs^{6a+8k-8j}rs^{-3a-4k+4j}rs^{6a+8k-8j}rs^{-3a-4k+4j}$$

so,

$$s^a rs^{6a+8k-8j}rs^{-a} = s^{4a+4k-4j}rs^{-6a-8k+8j}rs^{2a+4k-4j}.$$

We also know $rs^{6a+8k-8j}r \sim s^4$, so

$$\begin{aligned} s^a rs^{6a+8k-8j}rs^{-a} &= s^{4a+4k-4j}rs^{-6a-8k+8j}rs^{2a+4k-4j} \\ &= rs^{-6a-8k+8j}rs^{6a+8k-8j}. \end{aligned}$$

Now we consider the two possibilities for a modulo 4. If $a \equiv 1 \bmod 4$ then $a+1 \equiv 2 \bmod 4$ and

$$\begin{aligned} rs^{6a+8k-8j}r &= s^{a+1}rs^{-6a-8k+8j}rs^{-a-1}, \\ s^{-1}rs^{6a+8k-8j}rs &= s^a rs^{-6a-8k+8j}rs^{-a}. \end{aligned}$$

This implies

$$s^{-1}rs^{6a+8k-8j}rs = s^a rs^{-6a-8k+8j}rs^{-a} = rs^{-6a-8k+8j}rs^{6a+8k-8j} = y^{-1}x.$$

If $a \equiv 3 \bmod 4$ then $a + 1 \equiv 0 \bmod 4$ and

$$\begin{aligned} rs^{6a+8k-8j}r &= s^{a+1}rs^{6a+8k-8j}rs^{-a-1}, \\ s^{-1}rs^{6a+8k-8j}rs &= s^a rs^{6a+8k-8j}rs^{-a}, \end{aligned}$$

and so

$$s^{-1}rs^{6a+8k-8j}rs = s^a rs^{6a+8k-8j}rs^{-a} = s^{-6a-8k+8j}rs^{6a+8k-8j}r = x^{-1}y.$$

Hence $N = \langle x, y\rangle$. Also, $x^4 = 1$, $y^4 = 1$, $x^2 = y^2$ since $s^{12a+16k-16j}$ is central, and $x^{-1}yx = y^{-1}$ by (40), so N is a homomorphic image of Q_8. It remains to show that $N \cong Q_8$. For this we note that adding $s^8 = 1$ to the relations of $F^{a,b,c}$ when $(a, b, c) = 1$ and $d = 4$ yields a small number of distinct homomorphic images. Using GAP (see [7]) we easily check that each of these homomorphic images has derived length 4 and so the corresponding $F^{a,b,c}$ has derived length at least 4. This completes the proof for $d = 4$.

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