COMPUTING WITH 4-ENGEL GROUPS

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Abstract

We have proved that 4-Engel groups are locally nilpotent. The proof is based upon detailed computations by both hand and machine. Here we elaborate on explicit computer calculations which provided some of the motivation behind the proof. In particular we give details on the hardest coset enumerations now required to show in a direct proof that 4-Engel p-groups are locally finite for $5 \leq p \leq 31$. We provide a theoretical result which enables us to do requisite coset enumerations much better and we also give a new, tight bound on the class of 4-Engel 5-groups. In addition we give further information on use of the Knuth–Bendix procedure for verifying nilpotency of a finitely presented group.

1 Introduction

A group G is said to be an n-Engel group if

$$[x, \underbrace{y, y, \ldots, y}_{n}] = 1$$

for all $x, y \in G$. Here $[x, y]$ denotes $x^{-1}y^{-1}xy$, and $[x, y, \ldots, y]$ denotes the left-normed commutator $\Big[\big[\ldots[x, y], \ldots\big], y\Big]$.

Questions about nilpotency of groups satisfying Engel conditions have been long considered and in 1936 Zorn proved that finite Engel groups are nilpotent. In [6] (where there is more historical background) we proved

Theorem 1.1 *4-Engel groups are locally nilpotent.*

This result came after a number of results pointing in this direction. Traustason [9] showed that if 4-Engel groups of exponent p are locally finite, then 4-Engel p-groups are locally finite. Vaughan-Lee [13] proved that 4-Engel groups of exponent 5 are locally finite, and it follows from this and Traustason's work that 4-Engel 5-groups are locally finite. Then Traustason [11] proved that 2-generator, 4-Engel groups are nilpotent.

[1]The first author was partially supported by the Australian Research Council.

We started by aiming to extend the results of Vaughan-Lee [13], which showed that 4-Engel 5-groups are locally finite, to p-groups for other primes. Vaughan-Lee's proof uses the p-quotient algorithm and coset enumeration. Here we give details on a key step in proving local finiteness for primes p up to 31.

These results convinced us that the general result is true and provided enough motivation for us to carry through our general proof. The proofs for general 4-Engel groups and for 4-Engel p-groups follow the same line of argument as is used in [13] for 5-groups. Given our general proof and other results, here we focus on one particular part of a direct proof for small p.

2 4-Engel 5-groups

The proof for 4-Engel 5-groups is based on the following theorem which is proved in [13].

Theorem 2.1 *4-Engel groups of exponent 5 satisfy the identical relation*

$$[x,[y,z,z,z],[y,z,z,z],[y,z,z,z]] = 1.$$

This means that it suffices to consider 3-generator groups. The proof in [13] relies on a number of coset enumerations, and on a number of calculations with the p-quotient algorithm in 2- and 3-generator groups. Enough information is given to allow verification of the computations. In view of Traustason's result that 2-generator, 4-Engel groups are nilpotent and the straightforward nature of the p-quotient calculations, we focus on the most challenging 3-generator coset enumeration and its analogue for general p.

The computer tools that we use include the computer algebra systems GAP [2] and MAGMA [1]. We use the ACE coset enumerator (Havas and Ramsay [4]) either as available in GAP or MAGMA, or as a stand-alone program for some more difficult cases. For an up to date description of coset enumeration see Sims [8]. We need to complete various coset enumerations. To do so we investigate presentations in a similar way that other problems are addressed in [3]. Enumeration strategies are discussed in detail in [5] but not considered in this paper. Here we simply use an enumeration strategy which worked well enough in practice.

In Lemma 9 of [13] it is shown that the subgroup $\langle uv, vw\rangle$ has finite index, 5^4, in the group $G = \langle u, v, w \mid R_1\rangle$ where

$$R_1 = \begin{cases} u^5 = v^5 = w^5 = 1 \\ (u^r v^s w^t)^5 = 1 \text{ for } r = 1,2 \text{ and } s,t = \pm 1, \pm 2 \\ [u,v] = 1 \\ [w,u,u] = [u,w,w,w] = 1 \\ [v,w,w,w] = [v,w,w,v] = [v,w,v,w] = [v,w,v,v] = 1 \\ [w,u,w,v,v] = 1 \end{cases}$$

The coset enumeration is moderately difficult. Advances in coset enumeration mean that more recent enumerators define a total of 136926 cosets to complete the enumeration, in comparison to 276037 (with the same MAGMA code shown in [13] but using Magma V2.11, which incorporates an updated coset enumerator, instead of Magma V1.01). Notice that here we impose a collection of 5-th powers and some commutator relations, but no explicit 4-Engel relations. (We give actual coset enumeration performance figures throughout this paper, but be aware that the figures depend critically on exact details of presentations and enumeration strategies used. It should not surprise if related computations differ in statistics such as total cosets.)

3 4-Engel p-groups; theory

The proof for 5-groups can be extended to other p-groups by following a similar line of argument. In view of our result that 4-Engel groups are locally nilpotent we omit most of the details. Briefly, for an analogous sequence of groups we need to show that specific subgroups have finite index. The corresponding problem which we need to solve to obtain the analogue of Lemma 9 for general p requires us to replace 5-th powers in R_1 by p-th powers.

Our initial attempts at this were unsuccessful and it is instructive to understand why we failed. We started with the same kinds of relations, adding extra p-th powers, but the coset enumerations did not complete.

We were trying to enumerate the cosets of the subgroup $\langle uv, vw\rangle$ in the group $G = \langle u, v, w \mid R,\ \text{exponent } p\rangle$ where

$$R = \begin{cases} [u,v] = 1 \\ [w,u,u] = [u,w,w,w] = 1 \\ [v,w,w,w] = [v,w,w,v] = [v,w,v,w] = [v,w,v,v] = 1 \\ [w,u,w,v,v] = 1 \end{cases}$$

For $p = 7$ we tried various sets of 7-th powers in an effort to obtain the hypothetical index, 7^4, always without success. We were in fact studying the group

$$H_p = \langle u, v, w \mid R,\ \text{exponent } p\rangle$$

instead of the group

$$G_p = \langle u, v, w \mid R,\ \text{exponent } p,\ \text{4-Engel}\rangle.$$

The largest nilpotent quotient of G_p has class 4 and order p^{12} for $p \geq 5$. The same is true for H_5, but for H_7 it has class 6 and order 7^{17} and for H_p, $p \geq 11$, it has class 7 and order p^{19}. Our results [6] and those of Traustason [12] already imply that G_p is indeed nilpotent, and a tough coset enumeration can show that the same is true for H_5. The subgroup $\langle v, w\rangle$, which is easily shown to have order 5^5, can be shown to have index 5^7 using the presentation $\langle u, v, w \mid R_1\rangle$ of [13]. That enumeration uses a total of 122623883 cosets, but no doubt could be done much better using other presentations for preimages of H_5. However we do not know whether H_p is nilpotent or not for $p \geq 7$.

This realization led us to use extra relations in addition to exponent relations for $p = 7$. Whereas exponent 5 in the context of R was adequate for our purposes, this was not so for $p = 7$, the next case. The initial solution, once found, is easy. Impose only three p-th powers, but also impose extra 4-Engel relations. That we need only three p-th powers is a consequence of the following theorem.

Theorem 3.1 *Let G be a 4-Engel group generated by elements of prime order p, where $p \geq 5$. Then G has exponent p.*

Proof We know from Traustason's result [11] that 2-generator, 4-Engel groups are nilpotent. The nilpotent quotient algorithm then shows that 2-generator, 4-Engel groups have class at most 6. Now let G be a 4-Engel group generated by elements of order p for $p \geq 5$. To show that G has exponent p we need to show that if $a, b \in G$ have order p then $(ab)^p = 1$. The remarks above show that the subgroup $\langle a, b\rangle$ has class at most 6, and so if $p \geq 7$ then $\langle a, b\rangle$ is a regular p-group, and has exponent p. In particular $(ab)^p = 1$. When $p = 5$ the following simple MAGMA program shows that $(ab)^p = 1$.

```
G := Group<a, b | a^5, b^5>;
Q := NilpotentQuotient(G, 7: Engel := 4);
(Q.1*Q.2)^5;
```

□

Remark 3.2 The argument used in the proof above shows that 4-Engel p-groups are regular for $p \geq 7$. It is of interest to note that 4-Engel 5-groups are also regular. It is easy to verify using the nilpotent quotient algorithm that in a 4-Engel group

$$\begin{aligned}(ab)^5 &= a^5b^5[b,a]^{10}[b,a,a]^{10}[b,a,b]^{30}[b,a,a,a]^5[b,a,b,a]^{35}[b,a,b,b]^{35}\\ &\quad [b,a,b,a,a]^{10}[b,a,b,a,b]^5[b,a,b,b,a]^{170}[b,a,b,b,a,a]^{330}.\end{aligned}$$

These and related considerations enable us to provide tight bounds on the class of 4-Engel 5-groups. Corollary 2 of [6] states that if G is an m-generator, 4-Engel group then G is nilpotent of class at most $4m$, and if G has no elements of order $2, 3$ or 5 then G is nilpotent of class at most 7. Given our result that 4-Engel groups are locally nilpotent, this corollary follows from Traustason's work [9, 10] on the class of locally nilpotent 4-Engel groups. But if we restrict our attention to p-groups then we can sharpen this corollary. Traustason's work shows that if $p > 5$ then a 4-Engel p-group has class at most 7. He also shows that in a 4-Engel 3-group the normal closure of an element is nilpotent of class at most 3, and this implies that an m-generator, 4-Engel 3-group has class at most $3m$. The bound of $4m$ in Corollary 2 of [6] comes from Traustason's result that in locally nilpotent 2-groups and 5-groups the normal closure of an element is nilpotent of class at most 4. This bound is sharp, since in the free 4-Engel group of rank 3 the element $[x, y, x, x, z, x]$ has order 10. Despite this, it turns out that we can do better than $4m$ in the case of 5-groups.

Theorem 3.3 *Let G be an m-generator, 4-Engel 5-group. If $m = 2$ then the class of G is at most 6, if $m = 3$ then the class of G is at most 8, and if $m > 3$ then the class of G is at most $2m$. Furthermore, these class bounds are attained if G is the free 4-Engel group of exponent 5 and rank m.*

Proof Newman and Vaughan-Lee [7] proved that the free 4-Engel group of exponent 5 and rank m has class 6 if $m = 2$, class 8 if $m = 3$, and class $2m$ if $m > 3$. They first established that the associated Lie rings of free 4-Engel groups of exponent 5 are free Lie rings in the variety of Lie rings determined by the following multilinear Lie identities

$$\begin{aligned} 5x &= 0, \\ \sum_{\sigma\in\mathrm{Sym}(4)} [x, y_{1\sigma}, y_{2\sigma}, y_{3\sigma}, y_{4\sigma}] &= 0, \\ \sum_{\sigma\in\mathrm{Sym}(4)} [y_{1\sigma}, x_1, x_2, y_{2\sigma}, y_{3\sigma}, y_{4\sigma}] &= 0, \\ \sum_{\sigma\in\mathrm{Sym}(4)} [y_{1\sigma}, x_1, x_2, x_3, y_{2\sigma}, y_{3\sigma}, y_{4\sigma}] &= 0. \end{aligned}$$

They then computed the class and the order of the free Lie ring of rank m in this variety. Now let G be an m-generator, 4-Engel 5-group, and let L be the associated Lie ring of G. We will show that $L/5L$ satisfies these 4 multilinear identities. So Newman and Vaughan-Lee's result shows that $L/5L$ has class at most 6 if $m = 2$, class at most 8 if $m = 3$, and class at most $2m$ if $m > 3$. But $L/5L$, L and G all have the same nilpotency class, and so this proves the theorem.

It remains to show that $L/5L$ satisfies these 4 identities. Clearly $L/5L$ satisfies the identity $5x = 0$. The other 3 Lie identities are direct consequences of the group identity $[x, y, y, y, y] = 1$ in G. This follows from Wall's theory of multilinear Lie relators in varieties of groups [14], but we can also see it directly. The identity $\sum_{\sigma\in\mathrm{Sym}(4)}[x, y_{1\sigma}, y_{2\sigma}, y_{3\sigma}, y_{4\sigma}] = 0$ comes from expanding the group commutator

$$[x, y_1y_2y_3y_4, y_1y_2y_3y_4, y_1y_2y_3y_4, y_1y_2y_3y_4]$$

and picking out the terms which involve all the elements y_1, y_2, y_3, y_4. Since G is a 4-Engel group, if $x, y_1, y_2, y_3, y_4 \in G$ we obtain the group relation

$$\prod_{\sigma\in\mathrm{Sym}(4)} [x, y_{1\sigma}, y_{2\sigma}, y_{3\sigma}, y_{4\sigma}] \in \gamma_6(G),$$

and hence the Lie identity

$$\sum_{\sigma\in\mathrm{Sym}(4)} [x, y_{1\sigma}, y_{2\sigma}, y_{3\sigma}, y_{4\sigma}] = 0$$

in the associated Lie ring of G.

Next, consider the identity $[x_1x_2, y, y, y, y] = 1$ which holds in 4-Engel groups. Expanding, and using the fact that the normal closure of y in a 4-Engel group is nilpotent of class at most 4, we obtain

$$[x_1, y, y, y, y][x_1, y, x_2, y, y, y][x_2, y, y, y, y] = 1,$$

which gives the identity

$$[x_1, y, x_2, y, y, y] = 1.$$

We obtain the Lie identity

$$\sum_{\sigma\in\mathrm{Sym}(4)} [y_{1\sigma}, x_1, x_2, y_{2\sigma}, y_{3\sigma}, y_{4\sigma}] = 0$$

by substituting $y_1y_2y_3y_4$ for y, and expanding as above.

We similarly obtain the Lie identity

$$\sum_{\sigma\in\mathrm{Sym}(4)} [y_{1\sigma}, x_1, x_2, x_3, y_{2\sigma}, y_{3\sigma}, y_{4\sigma}] = 0$$

from the group identity

$$[x_1x_2x_3, y, y, y, y] = 1.$$

□

4 4-Engel p-groups; coset enumerations

0 Initially we started with the 24 4-Engel relations

$$[u, x, x, x, x], [v, x, x, x, x], [w, x, x, x, x]$$

where $x = u^{\pm 1}v^{\pm 1}w^{\pm 1}$. Using these relations together with u^p, v^p, w^p and R to give a presentation ${}_{24}G_p$ for a preimage of G_p we can readily enough find that $\langle uv, vw\rangle$ has index p^4 in G_7 by coset enumeration, in a total of 1631060 cosets. A Magma program to do this kind of computation is in Appendix A.

With a limit of 6 million cosets, the equivalent enumerations work for $p = 11$ and $p = 13$, but overflow for greater primes. For more information see Table 1. The successful enumerations in this Table for ${}_{24}G_p$ are already enough to suggest that the result holds for all p.

However, having succeeded with these enumerations, we can both simplify and generalize them. What we really are trying to do here is to find preimages of

$$\langle u, v, w \mid R,\ \text{exponent } p,\ \text{4-Engel}\rangle$$

in which $\langle uv, vw\rangle$ can be shown to have index p^4 by coset enumeration and with correct maximal p-quotient.

The p-quotient part of this is easy. Determining how to solve coset enumeration problems well is more an art than a science. Two major issues arise: finding appropriate presentations and finding appropriate enumeration strategies.

In the present case we found better presentations for our purposes which also revealed a surprise. Start with the 35 relations comprising u^p, v^p and w^p together with R plus the 24 4-Engel relations above and simplify them by using Tietze transformations (readily done using either GAP or Magma; we give a Magma program which does this and investigates the presentations in Appendix B). Then take the first 19 relations obtained this way (comprising 3 power relations, 8 consequences of R and 8 consequences of 4-Engel relations) to give a presentation for a preimage, ${}_8P_p$, of ${}_{24}G_p$. The group defined by ${}_8P_p$ still has the correct maximal

Table 1. Coset enumeration performance: ${}_{24}G_p$ and ${}_8P_p$

		Total cosets		cpu seconds	
p	Index	${}_{24}G_p$	${}_8P_p$	${}_{24}G_p$	${}_8P_p$
7	2401	1631060	127028	86	3
11	14641	2058258	632081	123	13
13	28561	5410257	2461035	339	51
17	83521	48875422	12201625	3720	325
19	130321	115063303	23231584	8781	702
23	279841	356330189	83165369	18574	2450
29	707281	$>$ 500000000	370372846		7383
31	923521	$>$ 500000000	484110102		10876

p-quotient but the presentatation is much better for coset enumeration as shown in Table 1. The choice of 8 relators on top of the power relators and R was made somewhat arbitrarily: it worked relatively well. Not only were the total numbers of cosets significantly better for ${}_8P_p$ than for ${}_{24}G_p$ but in addition the cpu time improvements were even greater. Because the presentations are shorter, processing time per coset is reduced. Thus, for $p = 23$ the enumeration took 2450 versus 18574 cpu seconds on a SparcV9 1200MHz processor.

Experiments were undertaken with fewer extra relators revealing interesting results for $5 \le p \le 31$. We denote the simplified presentation obtained from the 3 p-th powers and R plus the first (in the order given by our MAGMA code) i 4-Engel consequences by ${}_iP_p$. Naturally enough we started with $p = 7$, the first unknown case at the time. Total cosets for successful enumerations were as follows — ${}_8P_7$: 127028; ${}_7P_7$: 123554; ${}_6P_7$: 117447; ${}_5P_7$: 173953; ${}_4P_7$: 175140; ${}_3P_7$: 1763201; ${}_2P_7$: 1664225; and the enumerations failed to complete for ${}_1P_7$ and ${}_0P_7$ with 400 million total cosets.

Further investigation showed that with $i = 3$ or more extra relations the maximal p-quotient of ${}_iP_7$ is the same as for G_7. However for ${}_2P_7$ the class of the maximal p-quotient went up by one to 5 and the order up by a factor of p to p^{13}. For ${}_1P_7$ the class is also 5 but this time the order is up by a factor of p^2. For ${}_0P_7$ (just the powers and R), the class is 7 and the order is p^{19}. However, even though the maximal p-quotients vary in class and size, in all cases the image of the subgroup $\langle uv, vw\rangle$ has index p^4 in the maximal p-quotient.

Having observed this we repeated the experiments for the other primes in our range. For all of them we discovered the same story as far as p-quotients and subgroup index in the p-quotients is concerned. But some surprises were revealed in the coset enumerations. For all $11 \le p \le 31$ and all $0 \le i \le 8$ the subgroup $\langle uv, vw\rangle$ has index p^4 in ${}_iP_p$ and can be found by coset enumeration, with performance figures given in Table 2. Notice that the total number of cosets generally decreases with decreasing number of extra relations. Extra 4-Engel relations hinder rather than help coset enumeration prove that the relevant subgroup has finite index. The enumeration for ${}_0P_{23}$ took 525 versus 2450 cpu seconds for ${}_8P_{23}$ on a SparcV9

Table 2. Coset enumeration performance: ${}_0P_p$ to ${}_8P_p$

Group	Total cosets					
	${}_iP_{11}$	${}_iP_{13}$	${}_iP_{17}$	${}_iP_{19}$	${}_iP_{23}$	${}_iP_{29}$
${}_8P_p$	632081	2461035	12201625	23231584	83165369	370372846
${}_7P_p$	601107	2275056	11613947	25005869	64094145	294718450
${}_6P_p$	563866	2093938	10921180	20777684	70388182	268403697
${}_5P_p$	526937	2043525	10121438	19799079	62701875	227600531
${}_4P_p$	489386	1885669	9602477	18768587	58598226	259035552
${}_3P_p$	473244	1773915	8936948	17420731	59346654	215577073
${}_2P_p$	440993	1611026	8252655	17767232	48378123	189715507
${}_1P_p$	398821	1492580	7883161	15219869	52330359	186096576
${}_0P_p$	369260	1384228	7299078	13783117	50234568	170431569

1200MHz processor. Finally, for $p = 31$ we have ${}_0P_{31}$ uses 289274269 cosets (3438 seconds); ${}_1P_{31}$, 332261302; ${}_2P_{31}$, 329376595; ${}_3P_{31}$, 358424103; ${}_4P_{31}$, 397744988; ${}_5P_{31}$, 401683049; ${}_6P_{31}$, 425874010; ${}_7P_{31}$, 457785384; and ${}_8P_{31}$, 484110102 (10876 seconds).

In §3, we imposed extra 4-Engel relations to move from consideration of H_p to G_p. Now we see that we need to do this for the coset enumeration for $p = 7$, but not for larger primes. This leads us back to $p = 5$ where we find that ${}_{24}G_5$ uses 1332997 cosets; ${}_8P_5$, 261641; ${}_7P_5$, 259354; ${}_6P_5$, 256839; ${}_5P_5$, 253161; ${}_4P_5$, 250230; ${}_3P_5$, 247629; and the enumerations fail to complete for ${}_2P_5$, ${}_1P_5$ and ${}_0P_5$ in 400 million cosets.

We can draw the following conclusions from these computations. First and most important, we can prove a key step in a direct proof that 4-Engel p-groups are locally finite for $5 \leq p \leq 31$ by practical coset enumeration, but some care is needed to do so. (Note that the performance figures for $p = 5$ indicate that 5-th powers as used in [13] enable coset enumeration to work more easily than 4-Engel relations.)

There are a number of unresolved questions which arise. Our calculations end up showing that finitely generated 4-Engel p-groups are finite. However the status of some groups we investigated in the process is unclear. Is the group H_p finite for $p \geq 7$? Which groups constructed along the lines of the ${}_iP_p$ are finite? (Finiteness of 3-generator, 4-Engel p-groups implies that finite presentations like this do exist.) Why do the coset enumerations for ${}_1P_7$, ${}_0P_7$, ${}_2P_5$, ${}_1P_5$ and ${}_0P_5$ fail to complete?

5 Proving T nilpotent

The chronologically last step that we completed for our proof that 4-Engel groups are locally nilpotent is explained in [6, §4]. There we show that the group T presented by

$$\langle u, v, w \mid [u, v], [w, u, u, u], [w, u, u, w], [w, u, w, w], [w, v, v], [w, v, w], \text{4-Engel}\rangle$$

is nilpotent. Our proof used difficult computations with implementations of the Knuth–Bendix procedure. Subsequently, Traustason [12] found a very clever proof of the nilpotence of T which does not use the Knuth–Bendix procedure.

In our proof, we prefaced use of the Knuth–Bendix procedure by determining separately some additional relations which hold in T. Further study of the relevant computations, aided by suggestions from Charles Sims, reveals that we can use the Knuth–Bendix procedure to prove nilpotence without explicitly adding these extra relations.

Briefly, by adding more redundant generators and by altering the sequence of Knuth–Bendix iterations we can first obtain all of the required additional relations and subsequently deduce that T is nilpotent. This makes this part of the proof both shorter and faster.

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A MAGMA program for coset enumeration

The following program computes the index of $\langle uv, vw\rangle$ in ${}_{24}G_p$ and provides extra information including details of its maximal p-quotient.

```
// Edit the following line for different primes
p := 7;
G := Group<u, v, w | u^p, v^p, w^p,
// Commutator relations R
(u,v), (w,u,u), (u,w,w,w), (w,u,w,v,v),
(v,w,w,w), (v,w,w,v), (v,w,v,w), (v,w,v,v),
// 24 4-Engel relations
[ (u,x,x,x,x): x in
  [ u^e1 * v^e2 *w^e3 :e1 in [1,-1], e2 in [1,-1], e3 in [1,-1] ] ],
[ (v,x,x,x,x): x in
  [ u^e1 * v^e2 *w^e3 :e1 in [1,-1], e2 in [1,-1], e3 in [1,-1] ] ],
[ (w,x,x,x,x): x in
  [ u^e1 * v^e2 *w^e3 :e1 in [1,-1], e2 in [1,-1], e3 in [1,-1] ] ] >;

Q := NilpotentQuotient(G,8);
"Max NQ of G has class", NilpotencyClass(Q),
            "and order", FactoredOrder(Q);
H := sub<Q | Q.1*Q.2, Q.2*Q.3>;
"In the max NQ the subgroup image has index", FactoredIndex(Q,H);
H := sub<G | G.1*G.2, G.2*G.3>;
I,_,M,T := ToddCoxeter(G, H : CosetLimit:=6000000,
                       Strategy:=<1000,1>, SubgroupRelations:=1);
"Index", I,"/",M,"/",T ;
```

B MAGMA program for presentation simplification

The following program performs coset enumerations for the groups defined by the simplified presentations ${}_iP_p$ and provides extra information including details of maximal p-quotients and information on presentation lengths.

```
// Edit the following line for different primes
p := 7;
G := Group<u, v, w | u^p, v^p, w^p,
// Commutator relations R
(u,v), (w,u,u), (u,w,w,w), (w,u,w,v,v),
(v,w,w,w), (v,w,w,v), (v,w,v,w), (v,w,v,v),
// 24 4-Engel relations
[ (u,x,x,x,x): x in
  [ u^e1 * v^e2 *w^e3 :e1 in [1,-1], e2 in [1,-1], e3 in [1,-1] ] ],
[ (v,x,x,x,x): x in
  [ u^e1 * v^e2 *w^e3 :e1 in [1,-1], e2 in [1,-1], e3 in [1,-1] ] ],
[ (w,x,x,x,x): x in
  [ u^e1 * v^e2 *w^e3 :e1 in [1,-1], e2 in [1,-1], e3 in [1,-1] ] ]
>;
```

```
Q := NilpotentQuotient (G,9);
"Max NQ of G has class", NilpotencyClass(Q),
            "and order", FactoredOrder(Q);
H := sub<Q | Q.1*Q.2, Q.2*Q.3>;
"In the max NQ the subgroup image has index", FactoredIndex(Q,H);
"For G, plen and #rel", PresentationLength(G), #Relations(G);

for l in [11..19] do
   S := Simplify(G);
   "For", l, "rels: S, plen and #rel",
             PresentationLength(S), #Relations(S);
   d := Ngens (S);
   F := FreeGroup(d);
   r := Relations (S);
   srels := [LHS (x) * RHS (x)^-1: x in r];
   prels := [ F!Eltseq (x): x in srels[1 .. l] ];
   P := quo <F | prels>;
   "For P, plen and #rel", PresentationLength(P), #Relations(P);
   Q:=NilpotentQuotient(P,9);
   "Max NQ of P has class", NilpotencyClass(Q),
              " and order", FactoredOrder(Q);
   H := sub<Q | Q.1*Q.2, Q.2*Q.3>;
   "In the max NQ the subgroup image has index", FactoredIndex(Q,H);

   H := sub<P | P.1*P.2, P.2*P.3>;
   I,_,M,T := ToddCoxeter(P, H : CosetLimit:=6000000,
                          Strategy:=<1000,1>, SubgroupRelations:=1);
   "Index", I,"/",M,"/",T ;
end for;
```