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On Coxeter's families of group presentations

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ABSTRACT

In 1939 Coxeter published three infinite families of group presentations. He studied their properties, in particular determining when groups defined by members of the families are infinite and the structure of finite ones. Eight presentations remained for which the finiteness question was unsettled. We show that two of these eight presentations define finite groups (for which we give comprehensive proofs and provide detailed structural information) and that two of the presentations define infinite groups. Our results rely on substantial amounts of computer calculations, in particular on coset enumeration to prove finiteness and on computation of automatic structures using Knuth–Bendix rewriting to prove infiniteness.

1. Introduction

In 1939 Coxeter [3] considered the groups defined by the following presentations.

$$(\ell, m \mid n, k): \quad \left\{r, s \mid r^{\ell}, s^{m}, (rs)^{n}, \left(r^{-1}s\right)^{k}\right\},$$

$$(\ell, m, n; q): \quad \left\{r, s \mid r^{\ell}, s^{m}, (rs)^{n}, [r, s]^{q}\right\},$$

$$G^{m,n,p}: \quad \left\{a, b, c \mid a^{m}, b^{n}, c^{p}, (ab)^{2}, (bc)^{2}, (ca)^{2}, (abc)^{2}\right\}.$$

A recent paper by Edjvet and Juhàsz [4] provides a good overview of the history of investigations into these groups. Referring back to [14], it indicates that the finiteness problem has been resolved for

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all groups in the first family, $(\ell, m \mid n, k)$. The problem remained open for eight groups: $(2, 3, 13; 4)$, $(3, 4, 9; 2)$, $(3, 4, 11; 2)$, $(3, 4, 13; 2)$, $(3, 5, 6; 2)$ and $(3, 5, 7; 2)$ from the second family; and $G^{3,8,13}$ and $G^{3,7,19}$ from the third family.

Our interest in these groups was raised when the group $(2, 3, 13; 4)$ arose as a quotient of a one-relator quotient of the modular group studied by Conder, Havas and Newman [2]. As a result we decided to investigate these eight remaining groups. We prove that $(2, 3, 13; 4)$ and $G^{3,8,13}$ are finite and determine their structure. We show that $(3, 4, 13; 2)$ and $(3, 5, 7; 2)$ are infinite.

We have applied the same kinds of methods which resolve these four groups to the other four Coxeter presentations, and also tried other computational approaches to them. We have no new insights on whether they are finite or infinite.

The online version of this article contains supplementary materials. These are complete listings of an annotated Magma [1] program and its output which form the basis of our proof.

2. The group (2, 3, 13; 4) is finite

The group $G = (2, 3, 13; 4)$ was studied by Holt and Rees [11]. Making use of earlier computer calculations by a number of people, they show that G has a finite quotient with order 358 848 921 600, and composition factors $L_2(25)$, $L_3(3)$, and C_2 13 times. We prove that G is actually finite, and isomorphic to that known quotient. We also present two immediate consequences of this result.

Our proof relies heavily on computer calculations. (Detailed descriptions of the relevant methods and applications are provided in [13,10].) In the process of proving the main result, we effectively provide in one place proofs of the results listed in [11]. By downloading the program, interested readers can verify our claims and can modify the program to investigate the group further. We include comments on some snippets of the program in this paper.

First, the group Q_{12} studied by Conder, Havas and Newman which motivated our investigation is $\langle x, y \mid x^2, y^3, u^{10}v^2uvuv^2\rangle$ where $u = xy$ and $v = xy^{-1}$. It is easy to see that this one-relator quotient of the modular group (with extra relator of length 34) is isomorphic to $H = \langle c, d \mid c^2, d^3, (cd)^{13}[c, d]^{-4}\rangle$ which is a central extension of G. So to determine the structure of Q_{12} we need to understand G.

Second, as already proved by Coxeter [3], $(2, 3, 13; 4)$ is a subgroup of index 2 in $G^{3,8,13}$. Thus determining the structure of G renders it straightforward to determine the structure of $G^{3,8,13}$.

We show that G is finite of order $358\,848\,921\,600 = 2^{20}3^45^213^2$. We show also that $H = \langle c, d \mid c^2, d^3, (cd)^{13}[c, d]^{-4}\rangle$ is finite of order $2|G|$. We start with G.

The group G is perfect, and (the output of) our program shows that the action of G on the cosets of an index 26 subgroup

$$S = \left\langle a, (ba)^2b^{-1}, b^{-1}(ab)^3ab^{-1}(ab)^2ab^{-1}ab\right\rangle$$

is isomorphic to a group P that is an extension of an elementary abelian subgroup of order 2^{12} by the simple group $L_3(3)$. This normal 2-subgroup is complemented in P, so P is a split extension $2^{12}{:}L_3(3)$.

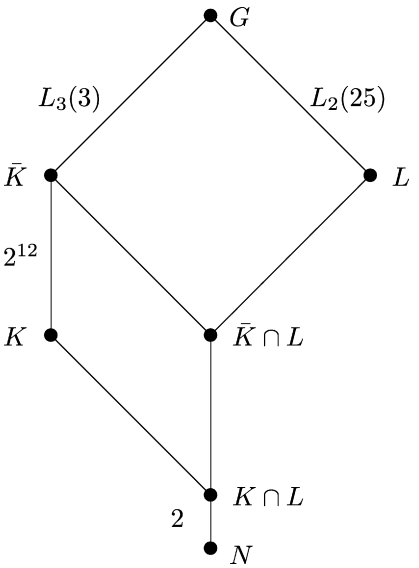
Our program continues by showing that the projection $\bar{h}$ of G onto $L_3(3)$ arising from this quotient can be defined as the action of G on the cosets of an index 13 subgroup, and that the kernel $\bar{K}$ of this projection has maximal 2-quotient which is elementary abelian with order 2^{12}. This proves also that the kernel K of the epimorphism $h_1 : G \to P$ has no proper 2-quotients.

The program shows next that G has an epimorphism h_2 onto $L_2(25)$ with perfect kernel which we denote by L. The maps h_1 and h_2 together define an epimorphism of G onto $P \times Q = 2^{12}{:}L_3(3) \times L_2(25)$ with kernel $K \cap L$.

The program shows now that the element $x := (ab)^2ab^{-1}$ lies in S and that its image in $P \times Q$ has the moderately large order $16 \times 13 = 208$. We plan to carry out a coset enumeration of $\langle x\rangle$ in S. Before proceeding to the coset enumeration, we compute the inverse image I of a Sylow 2-subgroup $P_2 \times Q_2$ of $P \times Q$ in G.

We then compute the largest 2-quotient of I, which has class 9 and order 2^{20}. So I has a quotient I/N with order 2^{20}, whereas $P_2 \times Q_2$ has order 2^{19}. Hence N has index 2 in $K \cap L$. Since I/N is the

largest 2-quotient of I, $(K \cap L)/N$ is the largest 2-quotient of $(K \cap L)$ and, since $K \cap L \lhd G$, this implies that $N \lhd G$. The quotient group G/N has structure $2.(2^{12}{:}\,L_3(3) \times L_2(25))$ and order $358\,848\,921\,600 = 2^{20}3^4 5^2 13^2$. The following diagram illustrates the quotients of G that we have found.



We saw above that the image of the element $x = (ab)^2 ab^{-1}$ in $P \times Q$ has order 208. The program now verifies that the order of its image in the larger quotient G/N is also 208. So the image $\langle x \rangle N$ of the subgroup $\langle x \rangle$ in S/N has index $|G/N|/(26 \times 208) = 66\,355\,200$.

We now show by coset enumeration that $\langle x \rangle$ has the same index 66 355 200 in S (see Proposition 5 for the rationale for this computation). The presentation that Magma calculates for S is on 3 generators, two of order 2 and one of order 3, and 7 relations, with total relation-length 92.

Since $N < S < G$, it follows also that the index of $\langle x \rangle$ in G is equal to the index of $\langle x \rangle N$ in G/N, and hence $N \leqslant \langle x \rangle$. Since G is perfect and N is cyclic, we must have $N \leqslant Z(G)$, so G is a perfect central extension of N by G/N. But G/N is itself a perfect central extension of $(K \cap L)/N$ by $P \times Q$ so, by the theory of perfect central extensions, G is a perfect central extension of $K \cap L$ by $P \times Q$.

By computing p-multipliers of P and Q for the relevant primes, we see that the Schur multiplier $M(P \times Q)$ of $P \times Q$ is elementary abelian of order 8 and, since N is cyclic, we have $|N| = 1$ or 2. But we have already shown that $(K \cap L)/N$ is the maximal 2-quotient of $K \cap L$, so we must have $N = 1$. Therefore G has the required order.

One remaining question is to determine the structure of the normal subgroup $\bar{K} \cap L$ of G of order 2^{13}, which could be either elementary abelian or extraspecial. Since the index $|I : \bar{K} \cap L| = 128$ is reasonably small, it is easy for the program to construct this subgroup and to verify that it is extraspecial. So G must be a central product of a group with the structure $2^{1+12}{:}\,L_3(3)$ and the group $2.L_2(25) \cong \mathrm{SL}(2, 25)$.

For any prime power p^{2n+1} with $n \geqslant 1$, there are two isomorphism classes of extraspecial groups of that order. For $p = 2$, these two types are represented by the central product of n copies of D_8 (the "+"-type) and the central product of $n-1$ copies of D_8 and one of Q_8 (the "−"-type). In our group G, the type of $\bar{K} \cap L$ is determined by the sign of the quadratic form that is preserved by the action of $G/\bar{K} \cong L_3(3)$ on the 12-dimensional module $\bar{K}/K$, and the program shows that $\bar{K} \cap L$ has "−"-type (using two different methods).

Turning now to the group H, we observe from the presentation that $G \cong H/Z$, where Z is the normal closure in H of the element $(cd)^{13}$. Furthermore, our program shows that H is perfect and is generated by cd and $[c, d]$. Since $(cd)^{13} =_H [c, d]^4$, the element $(cd)^{13}$ centralizes both cd and $[c, d]$, so $(cd)^{13} \in Z(H)$ and hence $Z \leqslant Z(H)$. So H is a perfect central extension of Z by G. From the structure

of G as a perfect central extension of $K \cap L$ by $P \times Q$ and the fact that $M(P \times Q)$ is elementary abelian of order 8, we can deduce that H is finite of order $k|G|$ for $k = 1, 2$ or 4.

Carrying out the same calculations for H as we did for G, which we do not discuss here in detail, we find that H has a quotient with the structure $2^{1+12}.L_3(3)$ and that H does not have a quotient with structure $2.L_2(25)$. The first of these facts shows that $|H| > |G|$, and the second shows that H is not isomorphic to the full covering group of $P \times Q$ (which *does* have a quotient $2.L_2(25)$). So $|H| = 2|G|$. As a consequence we have the following results.

Theorem 1. *The group* $(2, 3, 13; 4)$ *is finite with well-understood structure and order* $358\,848\,921\,600 = 2^{20}3^45^213^2$.

Corollary 2. *The group* $Q_{12} = \langle x, y \mid x^2, y^3, u^{10}v^2uvuv^2\rangle$, *where* $u = xy$ *and* $v = xy^{-1}$, *is finite with order* $2^{21}3^45^213^2$ *and is a central extension of* $(2, 3, 13; 4)$.

Corollary 3. *The group* $G^{3,8,13}$ *has* $(2, 3, 13; 4)$ *as a subgroup of index* 2 *so is finite with order* $2^{21}3^45^213^2$.

Further straightforward calculations in Magma show that $G^{3,8,13}$ has quotient groups isomorphic to the almost simple extensions $L_3(3).2$ and $\mathrm{PGL}(2, 25)$ of $L_3(3)$ and $L_2(25)$ respectively.

3. Commentary on our finiteness proof

The role of coset enumeration in our proof is evidenced by many of the Magma commands in our program, including `LowIndexSubgroups`, `Index` and `Rewrite` which rely on variations of the coset enumeration process. The critical new result is provided by the command:

```
Index(S, sub<S|x>: CosetLimit:=10^8,Hard:=true,Mendelsohn:=true);
```

We now explain how we chose to do that coset enumeration and also comment on other parts of our proof.

Once we observed that $G = (2, 3, 13; 4)$ is a quotient of the group Q_{12} we turned our attention to investigating G. Many attempts at proving it infinite which failed (now not surprisingly) are undocumented. At the same time we tried various coset enumerations in an effort to prove it finite. Naive attempts failed, but showed promising behaviour. (For example, incomplete coset enumerations in G over various subgroups using `Hard` strategies show that more than 1% redundant cosets are defined steadily during the enumeration process. This contrasts to similar enumerations in the known infinite groups $(2, 3, 7; 9)$ and $(2, 3, 7, 10)$, where much less redundancy is found. In $(2, 3, 7; 9)$ about 0.001% redundancies are seen; in $(2, 3, 7, 10)$ no redundant cosets at all are seen after the subgroup definition phase.)

The most common form of coset enumeration based proofs of finiteness rely on showing that a finite subgroup has finite index in the group. This may seem out of range for a group with order at least $358\,848\,921\,600$ and with no large finite subgroups visible from its presentation. However the proof method can be extended by using a result due to Schur [12, 10.1.4].

Proposition 4. *If* G *is a group whose centre has finite index* n, *then* G' *is finite and* $(G')^n = 1$.

This proposition leads to the following generalization of a result in [6].

Proposition 5. *A group is finite if its largest metabelian quotient is finite and it has a cyclic subgroup with finite index.*

Proof. Let G satisfy the hypotheses with $|G : C|$ finite and C cyclic. Let $K = \mathrm{Core}(C)$. Then $|C : K|$ is finite since $|G : C|$ is finite. Assume that K is infinite so $|\mathrm{Aut}(K)| = 2$. Hence $H = C_G(K)$ satisfies $|G : H| \leqslant 2$ and $K \leqslant Z(H)$. By Proposition 4, H' is finite. Now, since G/H' is metabelian hence finite

(as the largest metabelian quotient of G is finite), this implies G is finite (and our assumption is false). □

This enables us to use larger cyclic subgroups in coset enumerations to reduce the hypothetical index. We use our knowledge of finite quotients of G to find an element with large order.

Our proof starts off by explicitly constructing previously known finite quotients of G which are displayed in the diagram in Section 2. We can readily find elements with large order in these quotients by straightforward search. We do this by simply looking at short words in (the images of) the generators for words with maximal order in the quotients.

We find that $(ab)^2ab^{-1}$ has order 16 in the $2^{12}{:}L_3(3)$ quotient and order 13 in the $L_2(25)$ quotient. This means that it has order in G which is a multiple of $208 = 13 \times 16$. (In retrospect we can see that this is a shortest word with maximal order in G itself.)

This word has a fortunate property which makes our computations somewhat easier. It is in the subgroup S of index 26 in G that we have already computed. So instead of needing to try to compute the index of $\langle (ab)^2ab^{-1}\rangle$ in G it suffices to compute its index in S which (if finite) is smaller by a factor of 26. That is what we do in our proof in Section 2. In general, the space gain from carrying out a coset enumeration in a subgroup S of G rather than in G itself is paid for by increased processing time per coset, when the presentation for the subgroup is longer. In this example the computed presentation of S has three involutory generators and has moderately short total relator length, so the overall gains are significant.

Computing the inverse image I of a Sylow 2-subgroup $P_2 \times Q_2$ of $P \times Q$ is not completely straightforward in Magma. We use standard Reideister–Schreier rewriting based on coset tables. The index of $P_2 \times Q_2$ in G is 342 225, which is very large for computing a subgroup presentation, and Magma would have great difficulty doing this. Instead we do it very quickly using a two-stage process, going down by index 351 first, then index 975. This takes advantage of simplification after the first stage. Thus, after simplification, the subgroup $\mathtt{IP}_2$ of index 351 in G is presented with 4 generators, 23 relators, and relator length 990. The computed presentation of I, of index 975 in $\mathtt{IP}_2$ and 342 225 in G, which we do not simplify, has 2446 generators, 20 935 relators, and relator length 685 110. The p-quotient algorithm can be applied without undue difficulty to this presentation of I.

Now we provide a few comments on parameter selection for some commands in our program. Many commands in Magma have default behaviour but also have detailed parameter settings which vary the way the underlying algorithmic processes work. Sometimes the default settings are appropriate, but sometimes we choose to alter them.

By default the `Rewrite` command simplifies the computed subgroup presentation. This is generally appropriate behaviour and, in particular, is desirable if the subgroup presentation is to be used for coset enumeration. However it is not necessary (and sometimes even undesirable) if the subgroup presentation will be used only for `AQInvariants` or `pQuotient` computation. Thus we occasionally set `Simplify:=false`. We did this in the construction of the presentation of the subgroup I of S, for example, because Magma's attempts to simplify this presentation were unduly time-consuming and resulted in very long relators.

Coset enumeration based procedures have a very rich range of parameters. Selection methods to find good parameters are discussed in [8]. Suffice it to say, the parameters chosen for our hardest enumeration (`Hard:=true` and `Mendelsohn:=true`) enable the enumeration of the 66 355 200 cosets of $\langle x\rangle$ in S to be completed in a process which defines a maximum of 80 775 892 and a total of 112 162 109 cosets.

When our complete Magma program was run on an Intel Xeon E5430 CPU rated at 2.66 GHz, Magma reported that the total cpu time was 1550 seconds and that the total memory usage was 2563 MB. A breakdown of the computer time indicates that only two Magma commands took more than 1 cpu second: computing the maximal 2-quotient of I took 92 cpu seconds; and the index 66 355 200 coset enumeration took almost all of the cpu time, 1407 cpu seconds. As far as memory usage is concerned nothing apart from the large coset enumeration used more than 44 MB.

We were lucky that our element of large order in G was actually in its subgroup S. However we did not need this good fortune. Other high order elements can be found in subgroups with index 13. Indeed, the enumeration of the cosets of $\langle x\rangle$ in G can be completed using the standalone coset enu-

merator ACE3 [7]. (Current interface limitations prevent the definition of enough cosets to complete this enumeration in Magma, but it can be done with ACE3 via GAP [5] on machines with enough memory – at least 24GB of RAM.) The maximum number of cosets defined during the coset enumeration is equal to the index (1 725 235 200) and the total is only slightly more (1 780 966 535). Apart from the large number of cosets, this is an easy enumeration.

Since the GAP interface to ACE is intricate, we provide the following code which performs this computation (and the output provides some extra details about its working).

```
LoadPackage("ace"); TCENUM := ACETCENUM;; SetInfoACELevel(3);;
F2 := FreeGroup("a","b"); a:=F2.1;; b:=F2.2;;      z := ACEStats(
        [a,b], [a^2, b^3, (a*b)^13, Comm(a,b)^4], [(a*b)^2*a*b^-1]
        : workspace:="6G", mess:=10^8, hard, acecho, mendelsohn );
```

When this GAP program was run on an AMD Opteron 8360 SE CPU rated at 2.5 GHz, GAP reported that the cpu time for the coset enumeration was 3847 seconds.

4. The groups (3, 4, 13; 2) and (3, 5, 7; 2) are infinite

All but one of the groups proved infinite in [4, Lemmas 2.4, 2.5] were resolved computationally via Knuth–Bendix rewriting using the KBMAG package described in [9]. The method used is to construct and verify the correctness of an *automatic structure* for the group in question. This consists of a collection of finite state automata, one of which (the word-acceptor) accepts a unique word representing each group element (in fact the least such word under the shortlex/lenlex ordering). It is then straightforward to verify that the accepted language of this automaton, and hence also the order of the group, is infinite. The other automata in the automatic structure enable one to reduce arbitrary words in the group generators to their shortlex least representatives, and hence to efficiently solve the word problem in the group.

The same approach succeeds for the groups $(3, 4, 13; 2)$ and $(3, 5, 7; 2)$. Word acceptors and multiplier automata for these groups are available from the authors, and brief details follow.

The first of these was a moderately straightforward computation taking only a few minutes using KBMAG. The word acceptor in the automatic structure has 3147 states and the multiplier automaton has 7777 states. (This calculation can also be done using either the Magma or GAP interface to KBMAG.)

The example $(3, 5, 7; 2)$ is significantly more difficult. The automatic structure for this group was first computed by Alun Williams using his MAF package [15], and we are grateful to him for his assistance. His computation took about 6 hours on a Mac Pro and used about 2GB of memory. MAF has been shown to run faster than KBMAG on some of the more difficult examples. The word acceptor has 47 613 states and the multiplier 277 371 states, which constitute one of the largest automatic structures that have been computed to date. Since the bulk of the computation consists of preliminary constructions of automata that are much larger than those in the correct automatic structure that is eventually computed, it was straightforward to use KBMAG to confirm independently the correctness of Williams' calculations. We have also succeeded in computing the word acceptor and the word multiplier directly with KBMAG, although this took much longer.

Summing up, we have:

Theorem 6. *The groups* $(3, 4, 13; 2)$ *and* $(3, 5, 7; 2)$ *are infinite.*

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Supplementary material

The online version of this article contains additional supplementary material.
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