On one-relator quotients of the modular group

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Introduction

The modular group is a much studied object in mathematics.

Indeed in the documentation for the award of the 2009 Abel Prize to Mikhail Gromov, this group is described as
one of the most important groups in the modern history of mathematics

It is perhaps best known as the projective special linear group $L_2(\mathbb{Z})$, with a standard representation as a group of linear fractional transformations.

It is SQ-universal and has a large collection of interesting quotients, including most of the nonabelian finite simple groups.

We study the modular group as a finitely presented group; it is isomorphic to the free product of the cyclic groups C_2 and C_3, which gives its natural and shortest presentation: $\{x, y \mid x^2, y^3\}$.

What are the one-relator quotients of this group?

In other words, which groups can we obtain by adding one extra relator $w(x, y)$ to the standard presentation?

We are happy when we can either give the order of the group when finite or a proof of infiniteness otherwise.

If we can tell you the order, we can tell you its structure.

How can we determine the order of a finite FP-group?

Favorite tool: coset enumeration.

How can we prove a group to be infinite?

Simplest method:
it or one of its subgroups has an infinite abelian quotient.

More complicated methods: cohomology; geometry; graphical; rewriting (Knuth-Bendix); prove infinitely many different properties (eg, quotients); Golod-Shafarevich; Small Cancellation Theory, . . .

Or any of the above for a subgroup, quotient or section.

So, which groups can we obtain by adding one extra relator $w(x, y)$ to the standard presentation?

Many interesting cases are already known, including infinite sequences.

Work on Schur multipliers gives a necessary condition for a finite 2-generator group to be presentable with 3 relators.

Suffice it to say that a finite $\{2, 3\}$-generated group is not presentable as a one-relator quotient of the modular group unless rank$(M(G)) = 0$ or 1.

One-relator quotients of the modular group have been much considered over time.

Indeed, in 1856 Hamilton produced what we can read as a presentation for A_5 (which he called "Icosian Calculus") via a one-relator quotient of the modular group.

In 1901 G.A. Miller identified the triangle groups $\langle x, y \mid x^2, y^3, (xy)^n \rangle$ for $n = 2, 3, 4$ and 5, referring to Burnside (1897).

In 1902 he showed that they are infinite for $n > 5$.

In 1987 Conder substantially identified all one-relator quotients of the modular group defined using inequivalent extra relators with length up to 24.

There are 71 isomorphism types among those quotients which come from 300 presentations (34 distinct finite orders and 24 infinite).

(Subsequently Ulutaș and Cangül (2004) wrote a paper on this topic, but sadly their work is neither comprehensive nor fully correct.)

Generalized triangle groups: $w^n, n > 1$: 8 finite (Baumslag, Morgan, Shalen 1987; Howie, Metaftsis, Thomas 1995; Lévai, Rosenberger, Souvignier 1995)

Finite simple groups $PSL(2, m)$: − for prime $m \geq 5$
(Campbell and Robertson, 1980; $2m + 5$ syllables $u = xy$, $v = xy^{-1}$):

$$u^2vu^{(m-3)/2}vu^2vu^{(3m-3)/2}v$$

Finite soluble groups (Campbell, Heggie, Robertson and Thomas 1992):

$$xy^{-1}xy(xyxy^{-1})^{n-1}xy^{-1}xy$$

$G(n)$ is metabelian with order $6(n^2 + 3)$ if $n \not\equiv 3 \bmod 4$
and has derived length 3 and order $12(n^2 + 3)$ if $n \equiv 3 \bmod 4$.

The trivial group : ? how many ? (lots)

Importantly, many successful investigations (dating from 1980) into efficient presentations for simple groups have specifically studied one-relator quotients of the modular group.

To better understand the nature of one-relator quotients of the modular group, we extend Conder's 1987 work by investigating longer presentations.

We describe a canonical form for them.

In that context, we study all such quotients with extra relator having length up to 36, and determine the order of almost all of them.

Up to equivalence, there are 8296 presentations where the extra relator has length greater than 24.

We confirm Conder's 1987 results and we determine the order of all of the groups, except for six with length 36.

The presentations of the groups whose order we have not been able to determine provide interesting challenge problems.

When we can determine the order of a finite group, we are able to give detailed structural information about it.

Most of our results are based on computer calculations, which are sometimes substantial.

We use Magma, which provides excellent facilities for our needs.

(Alternatively GAP can be used to do the required computations.)

We provide supplementary materials, including some Magma programs on my website, together with their outputs.

http://www.itee.uq.edu.au/~havas/orqmg/

These programs and outputs give further details on our calculations and also provide information on computer resource usage.

Conder's approach revisited

Conder (1987) was motivated by a problem about graph embeddings to study what he called at the time "three-relator quotients of the modular group".

We now prefer the term *one-relator*, reflecting the count of extra relators, rather than the total relator count.

We define the modular group $\Gamma = \langle x, y \mid x^2, y^3 \rangle$ and consider its one-relator quotient $G = \langle x, y \mid x^2, y^3, w(x, y) \rangle$.

Any non-trivial element (other than x, y or y^{-1}) in Γ is conjugate to an element of the form $xy^{\epsilon_1}xy^{\epsilon_2} \ldots xy^{\epsilon_n}$ where $\epsilon_i = \pm 1$, which has n syllables and length $2n$.

We consider such elements as candidates for the extra relator.

There are 2^n of these, for n syllables.

We reduce the number to be considered by utilising automorphisms of the modular group.

We define $u = xy$ and $v = xy^{-1}$.

Then $u^{-1}v = y^{-1}x^{-1}xy^{-1} = y^{-2} = y$ and $vu^{-1}v = xy^{-1}y = x$, so Γ has alternative presentation $\{u, v \mid (vu^{-1}v)^2, (u^{-1}v)^3\}$, which we call P.

This presentation is more convenient for describing canonical representatives for the extra relator.

Taking the automorphisms of Γ into account reduces the number of presentations significantly.

Each relator we need to consider is a reduced word in u and v, and two such words are equivalent if one can be obtained from the other by cyclic permutation, reflection, or complementation (swapping u and v).

Hence the number of n-syllable relators is equal to the number of n-bead necklaces, where each bead is one of two colours, turning over is allowed, and complements are equivalent.

In particular, for syllable counts from 1 to 18 we obtain (in order) 1, 2, 2, 4, 4, 8, 9, 18, 23, 44, 63, 122, 190, 362, 612, 1162, 2056 and 3914 relators, matching the necklace count in given by Sloane:

`http://www.research.att.com/~njas/sequences/A000011`

$$\left(\sum_{d|n} \left(2^{n/d} \varphi(2d)/(2n) \right) + 2^{[n/2]} \right) /2$$

where ϕ is the Euler phi function.

The dominant term is $2^{n-2}/n$.

Our methods and the easy cases

We have implemented a Magma program which generates canonical representatives of extra relators with from 3 to 18 syllables, and tries to determine the order of the groups they define, using coset enumeration.

The most common form of proofs of finiteness based on coset enumeration rely on showing that a provably finite subgroup has finite index in the group.

The most common form of proofs of *infiniteness* based on coset enumeration rely on showing that a subgroup (with finite index) has infinite abelianisation.

We start by using coset enumeration over the trivial subgroup, allowing a maximum of $\mathbf{10^6}$ cosets.

If that fails, we look for subgroups with index up to 33 (in principle) and with infinite abelianisation.

The three presentations with extra relator having less than three syllables define finite groups.

For extra relator with from 3 to 18 syllables, we see that 8336 presentations define groups with explicit finite order, while 191 define infinite groups.

This leaves 66 groups with order to be determined, out of an initial total count of 8596.

A somewhat modified program (up to $\mathbf{10^7}$ cosets and subgroup index up to 42) reduces the number of outstanding cases to 48.

We find 11 more finite groups, and 7 infinite ones.

It is interesting to note that the two largest indices are found with quite easy coset enumerations, while some small index cases are quite difficult.

Commentary on the easy cases

Even though the methodology used thus far is both standard and relatively naive, it has proved to be very successful in determining the group order and even in addressing the isomorphism problem in most cases.

We progressively refine the techniques to address the harder problems.

Using the trivial group as subgroup in coset enumeration to prove finiteness is not ideal.

It has one implicit advantage, however, namely that when the enumeration succeeds, we get a regular representation for the group.

Given a representation for a permutation group of moderate size, it is straightforward to test its isomorphism with any group for which we have a permutation representation.

This means we have an implicit solution for the isomorphism problem among these finite quotients.

It is also easy to study group structure, eg, we can readily compute the normal subgroup lattice.

For the infinite groups, the information revealed about subgroups and sections by the Low Index Subgroups algorithm enables us to divide them into collections of distinct isomorphism types.

Output thus far reveals (inter alia):
trivial, $\mathbf{1856}$ presentations; $\boldsymbol{C_2}$, $\mathbf{2183}$; $\boldsymbol{C_3}$, $\mathbf{681}$; $\boldsymbol{C_6}$, 134; and $\boldsymbol{S_3}$, 799.

The largest two (thus far): orders $\mathbf{2\,359\,296}$ and $\mathbf{8\,491\,392}$.

Among the first $\mathbf{8336}$ finite groups found: $\mathbf{185}$ different orders, ??? isotypes.

We can readily determine the structure of all of these finite groups . . .

Small simple groups: all $\mathbf{43}$ order $\mathbf{60} - \boldsymbol{A_5}$; $\mathbf{14/53}$ order $\mathbf{168} - \boldsymbol{L_2(7)}$; $\mathbf{10/26}$ order $\mathbf{504} - \boldsymbol{L_2(8)}$; all $\mathbf{4}$ order $\mathbf{660} - \boldsymbol{L_2(11)}$; all $\mathbf{8}$ order $\mathbf{1092} - \boldsymbol{L_2(13)}$; all $\mathbf{9}$ order $\mathbf{2448} - \boldsymbol{L_2(17)}$; all $\mathbf{4}$ order $\mathbf{3420} - \boldsymbol{L_2(19)}$; and all $\mathbf{3}$ order $\mathbf{4080} - \boldsymbol{L_2(16)}$.

The smallest $\boldsymbol{\{2,3\}}$-generated simple group which does not occur is $\boldsymbol{L_3(3)}$, which has order $\mathbf{5616}$.

Its shortest presentations as a one-relator quotient of the modular group require (extra relators with) $\mathbf{21}$ syllables.

Also missing is $\boldsymbol{L_2(29)}$, having order 12180, whose shortest presentations require $\mathbf{19}$ syllables.

In contrast, both $\boldsymbol{L_2(31)}$, with order 14880 and three presentations, and $\boldsymbol{L_2(43)}$, with order 39732 and one presentation, do appear, with $\mathbf{17}$ syllables.

Note that the CR80 presentation for $\boldsymbol{L_2(43)}$ uses $\mathbf{91}$ syllables (the general presentation for $\boldsymbol{L_2(p)}$ uses $\boldsymbol{2p+5}$ syllables).

We have already seen infinite families of presentations that define an infinite number of different groups.

It is also easy to demonstrate infinite families of distinct presentations for the same group.

Theorem 1. *For each $n \geq 0$, $u^n v^{n+1}$ gives us the trivial group.*

Proof. Since $u^n v^{n+1} = 1$, the element $z = u^n = v^{-(n+1)}$ is central.
Conjugating by x gives $u^n = z = z^x = (v^{-(n+1)})^x = u^{n+1}$,
so $u = 1$, and the result follows. $\square$

Theorem 2. *For each $n > 0$, $u^n v$ gives us the cyclic group C_m, where $m = \gcd(n-1, 6)$.*

Proof. Since $v = u^{-n}$, the group is cyclic, and hence abelian.
The first two relations give $u^2 = v^4$ and $u^3 = v^3$,
from which it follows easily that $v = u^{-1}$ and $u^6 = 1$.
The third relation implies also $u^{n-1} = 1$, so the group has order $\gcd(n-1, 6)$. $\square$

The two families above are special cases of the more general family of groups with extra relator $u^n v^k$.

Here the element $u^n = v^{-k}$ is central, so the group is a central extension of $(2,3 \mid n,k) = \langle\, r,s \mid r^2, s^3, (rs)^n, (r^{-1}s)^k \,\rangle$, which is a member of the family of groups $(\ell, m \mid n, k)$ studied by Coxeter (1939).

Indeed, $(2,3 \mid n,k) \cong \langle\, r,s \mid r^2, s^3, (rs)^d \,\rangle$,
which is the $(2,3,d)$ triangle group, for $d = \gcd(n,k)$.

For $d = 1$ (when the triangle group is trivial), the central extension is perfect and so defines the trivial group if and only if $\gcd(n+k, 6) = 1$.

For other $d \le 5$, it defines a finite nontrivial group;
and for $d > 5$ it defines an infinite group.

Instances of other infinite families of presentations which define the trivial group can be observed.

One-relator quotients of the modular group that are trivial give rise to balanced presentations of the trivial group, and infinite families give rise to infinite families of balanced presentations.

By taking any presentation from one of these families, we can construct central extensions by amalgamating the relators $\boldsymbol{x^2}$ and $\boldsymbol{y^3}$.

In the case arising from Theorem 1, we may take $\boldsymbol{\{x, y \mid x^2y^3, (xy)^n(xy^{-1})^{n+1}\}}$, change any selection of $\boldsymbol{n+1}$ instances of $\boldsymbol{x}$ in the second relator into $\boldsymbol{x^{-1}}$.

Then the central extension is perfect, and hence trivial.

Each such presentation thus gives $\boldsymbol{\binom{2n+1}{n+1}}$ 'different' presentations of the trivial group.

In the case of Theorem 2, the group is trivial when $\boldsymbol{n \equiv 0}$ or $\boldsymbol{2 \bmod 6}$, and so we consider $\boldsymbol{\{x, y \mid x^2y^3, (xy)^n xy^{-1}\}}$.

Here, if $n \equiv 0 \bmod 6$ then we may change any $n/3$ instances of y to y^{-2} and any $n/2+1$ instances of x into x^{-1}, and obtain $\binom{n+1}{n/3}\binom{n+1}{n/2+1}$ presentations of the trivial group.

On the other hand, if $n \equiv 2 \bmod 6$, we may change any $(n-2)/3$ instances of y to y^{-2} and any $n/2$ instances of x to x^{-1}, and obtain $\binom{n+1}{(n-2)/3}\binom{n+1}{n/2}$ presentations.

Similar results hold for other one-relator quotients defining the trivial group.

Such one-relator quotients of the modular group can provide balanced presentations of the trivial group in numbers that are exponential in the presentation length.

These presentations (and variants based on presentations explicitly in terms of u and v instead of x and y) provide interesting candidates for counterexamples to the Andrews-Curtis conjecture.

Indeed, the examples from Theorem 1 correspond to variants of a family introduced by Akbulut and Kirby (1985).

Other examples, coming from trivial groups with extra relator of the form $u^n v^k$, seem to be new.

Harder presentations

Only 48 presentations remain, Q_i.

A quick perusal reveals that 12 presentations are for generalised triangle groups.

The order question has been resolved for all generalised triangle groups.

Nevertheless, we continue with our computational investigation of all these 48 presentations.

In our first attack on this collection of presentations, we applied our programs to the presentation of the group G on the initial generators x and y.

This is good for the Low Index Subgroups algorithm, as the presentation includes the order of the generators, but is not so good for standard coset enumeration; the presentation on generators u and v is better for Todd Coxeter enumerations because it is shorter.

We first attempted to prove infiniteness (as our methods of proving finiteness can waste resources if applied to infinite groups), by investigating subgroups and quotients more carefully.

Specifically, we looked at subgroups with index up to 42, the permutation representations afforded by their coset tables, and the abelian quotient invariants of both the subgroups and their cores (in cases where the core has index less than 2^{16}).

We found 11 more quotients that are infinite because they have subgroups with infinite cores.

In some other cases, we found very large abelianised cores and quotients which suggested that the groups may well be infinite.

Also some groups (11, only 2 covered above) clearly have quotients $L_2(p)$ for multiple values of p.

Work by Plesken and Fabianska (2009) has culminated in an algorithm for finding all quotients of a finitely presented group that are isomorphic to $L_2(q)$.

We have used an implementation of this algorithm due to Fabianska, and applied it to the groups listed above that have multiple $L_2(p)$ quotients.

This reveals that all 11 of those groups have L_2-quotients for infinitely many primes, and leaves 28 presentations to consider.

Initially we used coset enumerations over the trivial subgroup to prove finiteness, which had the benefit of giving us the group order directly and also giving us a regular representation for the group.

In some cases, however, we can do better by using a theorem of Schur on centre-by-finite groups which leads to the following known result.
Proposition 3. *A group is finite if its largest metabelian quotient is finite and it has a cyclic subgroup with finite index.*

This enables us to consider using larger cyclic subgroups in coset enumerations to reduce the hypothetical index, which leads to easier coset enumerations.

For each of our remaining 28 groups, the largest metabelian quotient is finite (since we know that all subgroups with index up to 6 have finite abelianisations).

We do not know a priori the orders of u and v (which are equal since $u^x = v^{-1}$), but we can perform coset enumerations over the subgroup generated by either of them.

Somewhat arbitrarily, we may choose v and try to enumerate the cosets of $\langle v \rangle$ in G, with the same maximum coset limit, namely 10^7.

We thus discovered 13 more finite groups, in which v has index: 292032; 78624; 110592; 3 538 944; 4; 172032; 1; 13; 367416; 1 572 864; 403368; 87500; and 5 308 416.

These groups are all finite, and given this knowledge, it is not too hard to determine their orders.

The index 3 538 944 one is a generalised triangle group, identified by LRS (1995) as a group with order $\mathbf{2^{20}3^{4}5 = 424\,673\,280}$.

The larger indexes here are for groups which are clearly out of range of our previous finiteness proof attempts.

By modifying our program to allow the definition of $\mathbf{10^{8}}$ cosets we found three more finite groups: indexes 746928; 31; and 712500.

This leaves 12 presentations (including just one generalised triangle group) to be resolved.

The standalone coset enumerator ACE3 allows the definition of more than $\mathbf{2 \times 10^9}$ cosets (avoiding current limits in the Magma implementation).

Using ACE3 we found that two more indexes: 63 824 112 and 36.

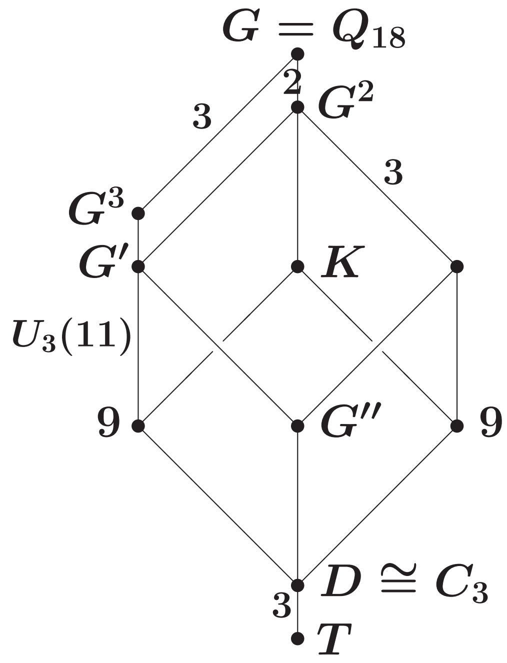
The best enumerations we have found use: a maximum of 309 366 526 and a total of 311 338 810 cosets; and a maximum of 948 327 123 and a total of 953 684 712.

These are very difficult enumerations.

When the index of $\langle \boldsymbol{v} \rangle$ is moderate we can determine the structure of the group reasonably easy.

$\boldsymbol{Q_{18}}$ with index 63 824 112 is more challenging, but we can determine its order (3 829 446 720) and structure using Magma (next slide):

We find that $\boldsymbol{Q_{18}}$ has $\boldsymbol{U_3(11)}$ as a section which enables us to construct nice presentations for $\boldsymbol{U_3(11)}$ in various ways.



For the 10 remaining presentations, perusal of the subgroup, quotient and section structure reveals that two of these are certainly very large.

One, $(u^3vuv^2)^2$, has sections (indeed cores with abelian quotient invariants) with orders $2^7 \times 6$, $5^6 \times 15$ and $3^7 \times 9$.

So far, our computational approach has not succeeded in proving it to be infinite; LRS (1995) use a cleverly constructed 3×3 matrix representation for its derived group.

The other, Q_{14} with extra relator $u^8vuvuv^2u^2v^2$, has sections with orders $2^6 \times 8^2$, 3^9 and 2×4^8.

Our computational approach has not succeeded in proving it to be infinite, but we can give an alternative proof.

In 1990 Conder studied a group related to trivalent symmetric graphs, which led to a two-relator quotient of the modular group and proved:
Proposition 4. $\langle x, y \mid x^2 = y^3 = (xy)^{12} = (xy^{-1}xy^{-1}xyxyxy^{-1}xy)^2 = 1\rangle$ *is infinite and insoluble.*

It is easy to see that it is a quotient of Q_{14}, and hence Q_{14} is infinite.

(Alternatively, information in Conder's paper enables us to build an 8×8 matrix representation for it, which directly demonstrates that it is infinite.)

That leaves 8 presentations — the status of the problem up to June 2009.

The last eight

The group Q_{12} with additional relator $u^{10}v^2uvuv^2$ is the only one with extra relator having less than 18 syllables that was not resolved above.

We see that the group has quotients $L_2(25)$ and $C_2{}^{12}.L_3(3)$.

Holt and Rees (1999) revealed these quotients (inter alia) in the group $(2,3,13;4)$, which is a member of another family of groups defined by Coxeter (1939): $(\ell, m, n; q) = \{r, s \mid r^\ell, s^m, (rs)^n, [r,s]^q\}$.

This observation leads us to note that Q_{12} is isomorphic to $H = \langle\, c, d \mid c^2, d^3, (cd)^{13}[c,d]^{-4} \,\rangle$ which is a central extension of $(2,3,13;4)$.

So, to determine the structure of Q_{12}, we need to understand $G = (2,3,13;4)$.

Coxeter's families of groups have been much studied since his paper was published.

A recent paper by Edjvet and Juhàsz (2008) provides a good overview of the history of investigations into them.

Suffice it to say, the order problem for G was unresolved as at the middle of 2009.

Motivated by our investigation here, Derek Holt and I decided to look at G again, and succeeded in proving that it is finite with order $\mathbf{358\,848\,921\,600}$.

The proof relies on coset enumeration, like our finiteness proofs here, but with careful cyclic subgroup selection to take advantage of a generator with as large order as possible.

We comprehensively described the structure of G, went on to show that Q_{12} has order $2|G| = 2^{21}3^{4}5^{2}13^{2}$, and described its structure.

$G = Q_{22}$ with extra relator $u^4vu^3v^3uv^4uv$ has simplest visible structure of the now seven remaining groups.

What we see is consistent with the hypothesis that this group is isomorphic to C_6, and for good reason: it is.

It was quite difficult, however, to prove this.

We tried many coset enumerations without success, each defining up to 2 billion cosets, in Q_{22} and in its index 2, 3 and 6 subgroups.

Another method for proving finiteness for finitely presented groups is Knuth-Bendix rewriting.

As a general rule, coset enumeration is much faster than Knuth-Bendix.

Sims (in his book, 1994) points out that Knuth-Bendix was able to find the order of a group defined by a presentation proposed by Bernhard Neumann as a challenge for computers, which at that stage no existing Todd-Coxeter implementation had handled.

Some other examples appear in HHKR (1999).

Neumann's example is resolved by coset enumeration in HR (2000), where there are further performance comparisons of coset enumeration and Knuth-Bendix rewriting.

It is easy using Reidemeister-Schreier rewriting to obtain a presentation for the derived group of $G = Q_{22}$, namely

$$G' = \langle a, b \mid bbABAbbaBBa, baaBAAABaab, babaBABABababA \rangle$$

(where $A = a^{-1}$ and $B = b^{-1}$ for ease of notation).

This was one of the presentations in which we attempted unsuccessful coset enumerations.

Experiments with Alun Williams' Monoid Automata Factory told us that G' is trivial.

Hence Q_{22} is isomorphic to C_6.

This proof is perhaps not entirely satisfactory, since one likely consequence of a bug in a Knuth-Bendix program is an incorrect total collapse.

So for additional reassurance, we have repeated the calculation using two independently written Knuth-Bendix implementations, Sims' RKBP and Holt's KBMAG (which is available via both GAP and Magma).

Both confirm the result.

This leaves six one-relator quotients of the modular group with extra relator of length 36 for which we are unable to determine finiteness or otherwise: $\boldsymbol{Q_{27}}$, $\boldsymbol{u^4vuvu^2v^2uvuv^4}$; $\boldsymbol{Q_{31}}$, $\boldsymbol{u^4vuv^4u^3vuv^3}$; $\boldsymbol{Q_{33}}$, $\boldsymbol{u^4v^2u^2v^4u^2vuv^2}$; $\boldsymbol{Q_{37}}$, $\boldsymbol{u^3vu^2vu^2v^2uv^2uv^3}$; $\boldsymbol{Q_{40}}$, $\boldsymbol{u^3vu^2v^2uv^3u^2vuv^2}$; and $\boldsymbol{Q_{43}}$, $\boldsymbol{u^3vuvuv^3u^2vuvuv^2}$.

We already have much information about all subgroups with index up to 42 in these groups, and about their cores.

There is enough to reveal that no two of these groups are isomorphic.

For example, the counts of the (conjugacy classes of) subgroups with index up to 42 are all different: 18, 23, 14, 12, 27 and 9, respectively.

We know that each of the groups has at least one $\boldsymbol{L_2}$-section:
$\boldsymbol{Q_{27} - L_2(19)}$; $\boldsymbol{Q_{31} - L_2(7)}$; $\boldsymbol{Q_{33} - L_2(13)}$; $\boldsymbol{Q_{37} - L_2(13)}$; and $\boldsymbol{Q_{40} - L_2(11)}$.

Looking more deeply at the subgroups with index up to six in these groups, we see that the index 3 subgroup of $\boldsymbol{Q_{43}}$ maps onto $\boldsymbol{L_2(64)}$ (as does its index 6 subgroup).

We know of only one other nonabelian simple section that occurs: the index 2 subgroup of $\boldsymbol{Q_{37}}$ maps onto $\boldsymbol{J_2}$ (as does its index 6 subgroup).

An easy computation enables us to show that each of the six groups has a largest soluble quotient and to compute its order.

We can also compute all normal subgroups with index up to 100000 and their abelian quotient invariants.

By multiplying the index of thus known normal subgroups by the orders of their abelianisations, we can compute lower bounds on the group orders.

We can also increase two of those bounds by multiplying them by the orders of independent sections, namely $L_2(64)$ for Q_{43} and J_2 for Q_{37}.

We obtain $|Q_{27}| \geq 430920$, $|Q_{31}| \geq 220\,814\,937\,504$, $|Q_{33}| \geq 124488$, $|Q_{37}| \geq 75\,290\,342\,400$, $|Q_{40}| \geq 5\,544\,000$, and $|Q_{43}| \geq 67\,616\,640$.

Concluding remarks

We have studied one-relator quotients of the modular group ‘in the small’, that is, with a short extra relator. It is interesting to compare and contrast our results with generic results.

Kapovich and Schupp (2009) have produced detailed information on random m-relator quotients of the modular group for all $m \geq 1$.

Their paper includes many interesting results, including: these quotients are generically essentially incompressible — that is, the smallest size of any possible finite presentation of such a group is bounded below by a function which is almost linear in terms of the length of a random presentation for it.

They also compute precise asymptotics of the number of isomorphism types of m-relator quotients where all the defining relators are cyclically reduced words of length n; and they obtain other algebraic results and show that such quotients are complete, Hopfian, co-Hopfian, one-ended, word-hyperbolic groups.

Earlier, Schupp (1976) proved that the triviality problem restricted to such presentations is undecidable.

The isomorphism problem for such presentations is thus certainly undecidable.

Indeed, Schupp's proof shows that the isomorphism problem restricted to certain fixed classes of such groups is undecidable.

On the other hand, rigidity shows that the isomorphism problem is generically easy.

We have shown that, in the small, most presentations of one-relator quotients of the modular group define finite groups.

We know that 220 out of 8596 with up to 18 syllables define infinite groups.

The finiteness question remains unresolved for six groups, and the rest are finite.

We can solve the isomorphism problem among these finite quotients, and expect that we can do the same for the infinite ones.

One consequence of the Kapovich and Schupp results is that as the relator length tends to infinity, almost all inequivalent presentations of one-relator quotients of the modular group define infinite groups.

This is very different to our results in the small.

Their Theorem C (Counting isomorphism types) specialises to give us a formula for $I(s)$, the number of isomorphism types of one-relator quotients of the modular group with s syllables.

Thus $\lim_{s\to\infty} I(s) = 2^{s-2}/s$, which is the dominant term in our count of inequivalent s-syllable presentations.

This (possibly surprising) result is consistent with one of Kapovich and Schupp's observations: "the first basic result is that a long random word over a finite alphabet is essentially its own shortest description."

We have six presentations out of 8596 for which we have not resolved the finiteness question.

One clear issue is that current computational methods for proving very large finitely presented groups to be finite are reaching their limits.

In particular, our lower bounds on the orders of Q_{31} and Q_{37} lead us to expect that a finiteness proof for either of them would be hard to find.

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