DIMACS Series in Discrete Mathematics
and Theoretical Computer Science
Volume **00**, 0000

Groups of deficiency zero

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Abstract. We make a systematic study of groups of deficiency zero, concentrating on groups of prime-power order. We prove that a number of p-groups have deficiency zero and give explicit balanced presentations for them. This significantly increases the number of such groups known. We describe a reasonably general computational approach which leads to these results. We also list some other finite groups of deficiency zero.

1. Introduction

In this paper we show how the use of symbolic computation changes the way in which one can attack previously intractable problems on group presentations.

The group defined by a finite presentation $\{X : \mathcal{R}\}$ is well-known to be infinite if $|X| > |\mathcal{R}|$. A group is said to have *deficiency zero* if it has a finite presentation $\{X : \mathcal{R}\}$ with $|X| = |\mathcal{R}|$ and $|Y| \leq |\mathcal{S}|$ for all other finite presentations $\{Y : \mathcal{S}\}$ of it. A presentation with the same number of generators and relators is called *balanced*. The *generator number* of a group G is the cardinality of a smallest generating set for G.

It was recognised quite early that groups of deficiency zero could be interesting – see, for example, Miller (1909). A recent account can be found in the lecture notes of Johnson (1990, Chapter 7). All known examples of finite groups of deficiency zero can be generated by at most 3 elements. All finite cyclic groups have deficiency zero. It is known precisely which metacyclic groups have deficiency zero. The first examples which cannot be generated by 2 elements were found by Mennicke (1959); others have been found by Wamsley (1970), Post (1978) and Johnson (1979). Only a very few of these examples are known to have prime-power order. There are two in Mennicke's list and eight in Wamsley's (of which two pairs are isomorphic). The theorem of Golod-Shafarevich (Johnson, 1990, Chapter 15) shows that a p-group with deficiency zero can be generated by at most 3 elements.

1991 *Mathematics Subject Classification.* Primary 20F05, 20D15; Secondary 20-04.
This paper is in final form, and no version of it will be submitted for publication elsewhere.

We will highlight several problems; some of these are well-known.

Problem 1. *Are all groups of deficiency zero which cannot be generated by three elements infinite?*

In this paper we focus attention mainly on p-groups and we exhibit more examples of p-groups with deficiency zero which have generator number 3. We do this by systematic and substantial use of implementations of algorithms. Access to these is provided via the computer algebra systems Cayley (Cannon, 1984), GAP (Schönert *et al.*, 1993) and Magma (Bosma & Cannon, 1993); the packages Quotpic (Holt & Rees, 1993) and the ANU p-Quotient Program (Newman & O'Brien, 1995); and various stand-alone programs. Our methods also yield some interesting information about 2-generator 2-relator presentations.

The Schur multiplicator provides a useful criterion in the search for finite groups of deficiency zero. It follows already from one of Schur's observations (see Johnson, 1990, p. 87) that a finite group of deficiency zero has trivial multiplicator. It is one of the outstanding questions about p-groups whether the converse holds; see, for example, Wamsley (1973, Question 12) and Johnson (1990).

Problem 2. *Exhibit a p-group with trivial multiplicator which does not have deficiency zero.*

The ANU p-Quotient Program can determine whether the multiplicator of a p-group is trivial. Thus we can search for groups of deficiency zero among the groups with trivial multiplicator. For a particular prime, we can find all such groups up to any (realistic) prescribed bound on the order. We prove that a p-group with generator number 3 and trivial multiplicator has order at least p^8. There are none of order 2^8 and 14 isomorphism types of order 3^8.

Knowing the groups turns out to be of little direct help in searching for balanced presentations. Instead we systematically generate appropriate balanced presentations and study these in some detail.

How can we decide whether such a presentation presents a group of interest? As is well-known there is no algorithm for deciding whether a finite presentation defines a finite group. However, if the presentation defines a finite group then there are procedures which will, in principle, prove this fact. Of these coset enumeration is generally the best in practice. Current implementations of coset enumeration can enumerate millions of cosets with reasonable resources; we use a derivative of that described by Havas (1991). However, it takes about 10 minutes on a Sparc Station 10/51 to enumerate 10 million cosets for the sorts of presentations we consider. In the light of this, we precede use of coset enumeration by faster tests which filter out presentations which have larger finite quotients than the ones being sought.

In practice, we calculate the p-quotient determined by each presentation to an appropriate class; if this quotient has the required order, we calculate the largest metabelian quotient; if this is also correct, we try to prove that the group is finite. This method has led to balanced presentations for 10 of the 14 groups of order 3^8. For the remaining groups we have balanced presentations which define them as pro-3-groups and as metabelian groups, but we have not been able to prove that they define the group.

Knowing the number of groups with trivial multiplicator of a given order provides us with a termination condition for our search. Another important feature of our approach is that it allows us to consider presentations which look unmanageable in more conventional approaches.

PROBLEM 3. *Prove the four remaining* 3*-generator groups with order* 3^8 *and trivial multiplicator have deficiency zero.*

PROBLEM 4. *Is there a* 3*-generator* p*-group of deficiency zero for any prime* p *other than* 2 *or* 3*?* (Wamsley, 1973, Question 21)

PROBLEM 5. *Are there infinitely many* 3*-generator* p*-groups of deficiency zero?*

We have applied the same techniques to 3-generator 2-groups and 5-groups, and 2-generator 5-groups and 7-groups. We give balanced presentations for the two 3-generator 2-groups of order 2^9 with trivial multiplicator. We also give balanced presentations for the six non-metacyclic 2-generator 5-groups with trivial multiplicator which have the minimal possible order, 5^5, and for the eight non-metacyclic 2-generator 7-groups with trivial multiplicator which have the minimal possible order, 7^5.

PROBLEM 6. *Are all groups of order* p^5 *with trivial multiplicator groups of deficiency zero?*

In Section 2 we investigate 3-generator p-groups with trivial multiplicator and describe a method which can be used to find all such groups of a given order. The resulting list contains all such groups of this order which have deficiency zero. In Section 3 we consider the general form that "short" presentations for such groups must take, discuss strategies to enumerate systematically lists of such presentations, and present new examples of deficiency zero groups. In Section 4 we present results for 2-generator groups.

We are indebted to Wei Ming Yan for help with some of the computations. All three authors were supported by the NSF to attend a workshop on "Computational Aspects of Geometric Group Theory" at the Geometry Center, University of Minnesota, where part of this work was carried out. The first author was partially supported by the Australian Research Council and the Australian National University.

2. 3-generator groups with trivial multiplicator

Let G be a p-group. The *lower exponent-p central series* of G is the descending sequence of subgroups

$$G = P_0(G) \geq \ldots \geq P_{i-1}(G) \geq P_i(G) \geq \ldots$$

where $P_i(G) = [P_{i-1}(G), G]P_{i-1}(G)^p$ for $i \geq 1$. If $P_c(G) = 1$ and c is the smallest such integer, then G has *class c*.

THEOREM 1. *A group which has a 3-generator 3-relator presentation and has a quotient which is elementary abelian of order p^3 for a prime p has a class 3 quotient of order p^8. If the group has order p^8, then its lower central quotients have orders p^3, p^3 and p^2.*

PROOF. Let F be a free group of rank 3. Then it is straightforward to see that $H := F/P_3(F)$ has order p^{23}. Let K be the Frattini subgroup of H. It is easy to see that K lies in the second centre of H, is abelian and has exponent p^2. Moreover for $h \in H$ and $k \in K$ the commutator $[k, h]$ has order dividing p. Hence the normal closure of k in H has order at most p^5. Thus, for $k_1, k_2, k_3 \in K$ the normal closure in H of $\{k_1, k_2, k_3\}$ has order at most p^{15}. Let G be a group of the kind given in the statement. Then $G/P_3(G)$ is a quotient of H by a normal subgroup generated by at most three elements of K. Hence $G/P_3(G)$ has order at least p^8.

If G has order p^8, then $P_3(G)$ is trivial and thus G is a quotient of H. Let N be the kernel of a map from F to G. Then $N \geq V := P_3(F)$. Since G has trivial multiplicator $[N, F] = N \cap F'$ and so $V[N, F] = V(N \cap F') = N \cap VF'$. Hence $N/V \cap H' = [N/V, H]$. Since H can be generated by 3 elements and N/V can be generated by 3 elements modulo the centre of H, it follows that $N/V \cap H'$ is an elementary abelian group with at most 9 generators. Therefore G' has order p^5 and G/G' has order p^3; it follows that $G'/\gamma_3(G)$ has order p^3 and $\gamma_3(G)$ has order p^2. □

Recall that a necessary condition for a p-group to have deficiency zero is that it have trivial multiplicator. We say that a group is a *candidate* if it has trivial multiplicator. We can get a list of candidates, since, for a given order, we have the practical tools available to construct one representative for each isomorphism type of group with trivial multiplicator. We now summarise the algorithm used to obtain these.

Let G be a finite p-group with generator number d and exponent-p class c. Given a finite presentation for G, we can compute a power-conjugate presentation for G using the p-quotient algorithm, which is described in Havas & Newman (1980) and implemented as part of the ANU p-Quotient Program. A group H is a *descendant* of G if H has generator number d and the quotient $H/P_c(H)$ is isomorphic to G. A group is an *immediate descendant* of G if it is a descendant of G and has class $c+1$. Given as input a power-conjugate presentation for the group

G, defined as a quotient, F/R, of the free group F on d generators, the p-group generation algorithm, described in Newman (1977) and O'Brien (1990), produces as output power-conjugate presentations for a complete and irredundant list of the immediate descendants of G. The *p-covering group*, G^*, of G is defined to be $F/[R, F]R^p$. Given a power-conjugate presentation for G, a power-conjugate presentation for G^* can be computed using the ANU p-Quotient Program. The *nucleus* of G is $P_c(G^*)$; if the nucleus is trivial, then G is *terminal*, otherwise G is *capable*. If G has order p^8 and trivial multiplicator, then its p-covering group has order p^{11} – this provides an easy criterion for recognising that the group has trivial multiplicator. Since these groups of necessity have trivial nucleus, they are terminal. Thus, we can use the p-group generation algorithm to generate a complete and irredundant list of presentations for the relevant p-groups and select those which have trivial multiplicator.

In the 3-generator p-group context, Theorem 1 gives that all groups of deficiency zero have order at least p^8; in addition, they have a class 2 quotient of order p^6, which has commutator subgroup of order p^3, and this quotient must have a nucleus of order at most p^2. We first use the p-group generation algorithm to generate descriptions of the relevant class 2 quotients and then reapply the algorithm to each class 2 quotient; the terminal groups obtained are our candidates. We used the p-group generation algorithm in the manner described to determine the smallest 3-generator p-groups with trivial multiplicator for $p = 2$, 3 and 5.

There are 10 3-generator groups of order 2^6 with commutator subgroup of order 2^3. Three have a nucleus of order 2^2 and all of the 15 immediate descendants of order 2^8 are capable. One of these has a nucleus of order 2, and has two terminal immediate descendants of order 2^9 which have trivial multiplicator. Four of the 10 have a nucleus of order 2^3, and all 60 immediate descendants of order 2^9 are capable. Hence there are no 3-generator groups of order 2^8 with trivial multiplicator and exactly two such groups of order 2^9.

Among the 16 groups of order 3^6 with commutator subgroup of order 3^3, five have a nucleus of order 3^2. Three of these each have 3 terminal immediate descendants of order 3^8; the fourth has 5 terminal immediate descendants; all immediate descendants of the fifth are capable. Hence there are 14 3-generator groups of order 3^8 with trivial multiplicator.

Among the 20 groups of order 5^6 with commutator subgroup of order 5^3, seven have a nucleus of order 5^2. Five of these each have 5 terminal immediate descendants of order 5^8; a sixth has 7 terminal immediate descendants; all immediate descendants of the seventh are capable. Hence there are 32 3-generator groups of order 5^8 with trivial multiplicator.

3. 3-generator groups of deficiency zero

3.1. Known examples. Mennicke (1959) lists the following presentations

$$M(\alpha,\beta,\gamma) \quad = \quad \{a,b,c \;:\; a^b = a^\alpha,\ b^c = b^\beta,\ c^a = c^\gamma\}$$

where we have rewritten the conjugate relators in their more usual form. He proved that the groups so presented are finite for most values of the parameters. Exactly two are p-groups: $M(3,3,3)$ and $M(-2,-2,-2)$ determine groups of order 2^{11} and 3^9, respectively.

Wamsley (1970) lists the following presentations:

$$\begin{aligned} G_1(\alpha,\beta,\gamma) \quad &= \quad \{a,b,c \;:\; a^c = a^\alpha,\ b^{c^{-1}} = b^\beta,\ c^\gamma = [a,b]\}, \\ G_2(\alpha,\beta,\gamma) \quad &= \quad \{a,b,c \;:\; a^c = a^\alpha,\ b^c = b^\beta,\ c^\gamma = [a,b]\}. \end{aligned}$$

In his list, there are four presentations for 2-groups: $G_i(3,3,2)$ and $G_i(3,3,-2)$ for $i = 1,2$. All four groups have order 2^{13}. How many different isomorphism types occur among these presentations?

O'Brien (1994) describes a practical algorithm which provides an answer to the problem for finite p-groups. He defines a *standard presentation* for each p-group and provides an algorithm which allows its construction. Hence given two p-groups presented by arbitrary finite presentations, the determination of their isomorphism is essentially the same problem as the construction of their standard presentations and the (trivial) comparison of these presentations. His implementation of this algorithm is available as part of GAP and MAGMA.

We used the standard presentation algorithm on the four Wamsley presentations to establish that there are just three isomorphism types, with $G_2(3,3,2)$ and $G_2(3,3,-2)$ presenting isomorphic groups.

The presentations $G_2(-2,-2,3)$ and $G_2(-2,-2,-3)$ determine groups of order 3^8. Each is isomorphic to the group presented by #5 in the list given in Section 3.4. The presentations $G_1(-2,-2,3)$ and $G_1(-2,-2,-3)$ are for distinct groups of order 3^{11}.

3.2. Direct approaches. Use of the p-group generation algorithm provides us with power-conjugate presentations for candidate groups. If a candidate has a 3-relator presentation, then a sequence of Tietze transformations exists which will convert its power-conjugate presentation to a 3-relator presentation. While the determination of a suitable sequence runs into unsolvability problems in general, some practical approaches exist which may succeed. Tietze transformation programs exist both in stand-alone versions (Havas, Kenne, Richardson & Robertson, 1984; Havas & Lian, 1994) and in Cayley, GAP, and MAGMA. The simplest idea is to take the power-conjugate presentation for a group, and to attempt to use such programs to find balanced presentations. In practice, this approach did not succeed for these groups.

An alternative is to try the (coset enumeration based) relation finding algorithm of Cannon (1973), which is available in Cayley. It is also conceivable that further "massaging" of these presentations – such as investigating 3-element subsets of the known relator sets – would lead to a positive outcome. This type of investigation has yielded efficient presentations in other contexts; see, for example, Kenne (1983). Neither of these approaches was successful here.

Since these direct methods do not give us balanced presentations, we instead search through a list of potentially suitable presentations.

3.3. What sort of presentations? The *exponent sum matrix* E of the presentation $\{a, b, c : u, v, w\}$ is the 3×3 matrix with columns labelled by the generators and rows by the relators and with $E(u, a)$ the exponent sum of a in u.

THEOREM 2. *If a group of order p^8 has a 3-generator 3-relator presentation, then it has a presentation $\{a, b, c : u, v, w\}$ where the length of each relator is at least $p + 2$ and the exponent sum matrix is diagonal with entries p, p, p.*

PROOF. We use the notation of Theorem 1. To get a presentation of the required form an appropriate sequence of Tietze transformations is applied to the given presentation. These are found by mirroring the steps of reducing the exponent sum matrix to diagonal form in such a way that the matrices are the exponent sum matrices of the corresponding presentations. Clearly permutations of rows and columns pose no problem and each row operation can be mirrored by elementary Tietze transformations. The operation of multiplying a column by -1 corresponds to the sequence of Tietze transformations which replaces the corresponding generator by its inverse; and the operation of adding the column labelled b to the column labelled a corresponds to the sequence which replaces the generator b by $a^{-1}b$. Since F'/NF' is elementary abelian of order p^3 the presentation can be transformed into a presentation whose exponent sum matrix is diagonal with entries p, p, p. Note that, in the proof of Theorem 1, if k is a p-th power then its normal closure has order at most p^4. So, finally, since the order of N/V must be p^{15}, no relator can be a p-th power and each relator must have length at least $p + 2$. □

We say that the *length* of a presentation is the sum of the lengths of the relators occurring in it. Thus Theorem 2 states that the shortest possible presentations for a group of deficiency zero have length $3p + 6$. We begin by considering these shortest presentations. Naturally we can reduce the search by eliminating presentations which are obviously equivalent to ones already considered. A relator of length $p + 2$ with exponent sum p in a is essentially of the form $b^{-1}a^iba^{p-i}$ with $i \in \{1, \ldots, p-1\}$. Moreover if two such relators both involve only two of the generators, and therefore the third involves the third generator and one of the original two, the group is infinite (factor out the generator which occurs in all three relators). Thus it suffices to consider the presentations:

$$\{a, b, c : b^{-1}a^iba^{p-i},\ c^{-1}b^jcb^{p-j},\ a^{-1}c^kac^{p-k}\}.$$

For $p = 2$ there is just one such presentation: $M(-1,-1,-1)$ which (as is well-known) defines an infinite group. For $p \geq 5$ factoring out one generator gives a metacyclic group which is not a p-group. For $p = 3$ all these presentations can be Tietze transformed into the one with $i = j = k = 1$ which is $M(-2,-2,-2)$.

We now use Theorem 2 to create a list of "short" presentations in length order for some specific primes. Since the list soon becomes very large, we first eliminate presentations which are obviously equivalent to ones already considered. We next "filter" from the list presentations which we can decide do not present the required groups. In practice, we want to use fast and cheap filters.

We can compute p-quotients of small class for a finitely presented group very rapidly. Since each of our candidate groups has class 3 and order p^8, the group given by a candidate presentation must have a largest p-quotient of order p^8. Hence we filter out all presentations which have a class 4 p-quotient with order larger than p^8. The ANU p-Quotient Program takes about 2 seconds to compute this information for 100 of the presentations considered for the prime 3 on a Sparc Station 10/51.

Moreover each of our candidate groups is metabelian; so we may further check that candidate presentations which pass the p-quotient test also have largest metabelian quotient of order p^8. This is readily done by computing the abelian quotient invariants of the commutator subgroup; here, we use coset enumeration (Havas, 1991), Reidemeister-Schreier subgroup presentation algorithms (Neubüser, 1982), and integer matrix diagonalization (Havas, Holt & Rees, 1993). This test is more expensive, taking about 10 seconds for 100 of the presentations considered for the prime 3 on a Sparc Station 10/51.

We next use the standard presentation algorithm to organise the candidate presentations into families according to the isomorphism type of the relevant p-quotient.

Only now do we apply coset enumeration to members of each family in order to attempt to establish that a surviving presentation does present a group of the correct order.

Recall that coset enumeration takes as input a group given by a finite presentation and a finitely generated subgroup of it, and, if the subgroup has finite index, gives as output the index. For our purposes we need to be able to find a finitely generated subgroup where the coset enumeration completes and where we can prove the subgroup finite. The simplest case is to take the trivial subgroup. The next best case is to take cyclic subgroups. If one can be found with finite index, then the Reidemeister-Schreier algorithm can be used to find a presentation for it. Its order can then be calculated using integer matrix diagonalization since the subgroup, being cyclic, is abelian. We have also been able to use 2-generator subgroups of finite index to complete finiteness proofs.

An alternative method for proving finiteness is based on the Knuth-Bendix procedure – see Sims (1994). We have not made a systematic attempt to use this tool for these problems. In private communication, Charles Sims reported that the Rutgers Knuth-Bendix package was able to prove the finiteness of $M(3,3,3)$ after a substantial computation.

3.4. 3-groups. Since $M(-2,-2,-2)$ is the only presentation of length 15, we go on to consider the presentations of length 17, where two relators have length 5 and one has length 7. Without loss of generality, we can consider the two relators of length 5 to be one of the following pairs:

$$\begin{array}{ll} a^2b^{-1}ab, & b^2c^{-1}bc \\ a^2c^{-1}ac, & b^2c^{-1}bc \\ a^2c^{-1}ac, & b^2cbc^{-1}\,. \end{array}$$

We now describe an algorithm which can be used to generate the set consisting of all relevant candidates for the relator of length 7. We may assume without loss of generality that the final relator of length 7 is freely and cyclically reduced. We consider each of the following sequences in turn:

$$\begin{array}{c} a, a^{-1}, a, a^{-1}, c, c, c \\ a, a^{-1}, b, b^{-1}, c, c, c \\ a, a^{-1}, c, c^{-1}, c, c, c \\ b, b^{-1}, b, b^{-1}, c, c, c \\ b, b^{-1}, c, c^{-1}, c, c, c\,. \end{array}$$

For each sequence, we construct those permutations which (when treated as a group word in the natural way) give words which freely and cyclically reduce to a word of length 7. From this set of words we take as a candidate relator one representative of each subset of cyclic permutations of the same word.

We use the natural generalisation of this technique to generate sets of relators of given length, where each of two generators occurs with exponent sum zero and the other occurs with exponent sum 3. Using this construction, we obtain 204 presentations of length 17.

To write down presentations of length 19, we first consider the case where one relator has length 5 and the other two have length 7. Here we can choose the relator of length 5 to be $a^2b^{-1}ab$. We now use the algorithm described above to write down first the set of relators of length 7 where c occurs with exponent sum 3, and then the set of relators of length 7 where b occurs with exponent sum 3. We also consider the case where two relators have length 5 and the third has length 9. We use the natural generalisation of the algorithm described above to write down the set of possible relators of length 9, where we first extend each of the five sequences of length 7 listed above to contain an additional generator-inverse pair. Using these constructions, we write down a total of 9 304 presentations.

Finally, we consider presentations of length 21 where all three relators have length 7. Here we use our algorithm to generate the three sets of relators of length 7 obtained by allowing each of a, b, c to occur with exponent sum 3. We obtain 82 688 presentations.

A total of 77 179 of the 92 196 presentations fail one of the filter tests. We classify the surviving candidates by the isomorphism type of their 3-quotient. Thirteen of the 14 isomorphism types of groups with trivial multiplicator occur among presentations of length at most 21. Nine of the 13 have presentations of length at most 19.

Do any of the remaining presentations determine groups of order 3^8? Coset enumeration over the trivial subgroup is sufficient to establish that the first five of the presentations listed below present groups of order 3^8. For the remaining presentations, coset enumerations do not complete over the trivial subgroup with a 10 million coset limit. However, coset enumeration over the subgroup generated by b allows us to conclude that presentations #6 to #9 describe groups of order 3^8. In each case, coset enumeration establishes that the cyclic subgroup has index 3^5 and integer matrix diagonalization shows that the subgroup has order 3^3.

For presentation #10 we have to work harder. Coset enumerations over the cyclic subgroups generated by the group generators do not complete, with a 10 million coset limit. Since a and b satisfy the relator $a^2b^{-1}ab$, the subgroup $\langle a, b\rangle$ is metabelian. We show that the subgroup has index 3^3, use the Reidemeister-Schreier algorithm to obtain a presentation for it, and then show that its metabelian quotient has order 3^5.

$$
\begin{aligned}
\#1.\ & \{a,b,c \ :\ b^2c^{-1}bc,\ a^2b^{-1}ab,\ ca^{-1}b^{-1}cabc\},\\
\#2.\ & \{a,b,c \ :\ b^2c^{-1}bc,\ a^2b^{-1}ab,\ b^{-1}abc^2a^{-1}c\},\\
\#3.\ & \{a,b,c \ :\ b^2c^{-1}bc,\ a^2b^{-1}ab,\ ca^{-1}c^2b^{-1}ab\},\\
\#4.\ & \{a,b,c \ :\ b^2c^{-1}bc,\ a^2b^{-1}ab,\ cab^{-1}cbca^{-1}\},\\
\#5.\ & \{a,b,c \ :\ b^2c^{-1}bc,\ a^2c^{-1}ac,\ a^{-1}b^{-1}abc^3\},\\
\#6.\ & \{a,b,c \ :\ a^2c^{-1}ac,\ bcac^{-1}b^2a^{-1},\ cbac^2b^{-1}a^{-1}\},\\
\#7.\ & \{a,b,c \ :\ a^2b^{-1}ab,\ ab^2cba^{-1}c^{-1},\ cb^{-1}acbca^{-1}\},\\
\#8.\ & \{a,b,c \ :\ a^2c^{-1}ac,\ acb^2a^{-1}c^{-1}b,\ a^{-1}b^{-1}abc^3\},\\
\#9.\ & \{a,b,c \ :\ b^{-1}abca^2c^{-1},\ acb^2c^{-1}a^{-1}b,\ cb^{-1}acbca^{-1}\},\\
\#10.\ & \{a,b,c \ :\ a^2b^{-1}ab,\ a^{-1}b^3c^{-1}ac,\ a^{-1}bc^2ab^{-1}c\}.
\end{aligned}
$$

We have not been able to decide whether any of the remaining four candidate groups of order 3^8 has a 3-relator presentation. We have various candidate presentations and we list one for each group. These have passed all of the filters.

$$
\begin{aligned}
\#11.\ & \{a,b,c \ :\ cac^{-1}b^{-1}aba,\ bacba^{-1}c^{-1}b,\ cb^{-1}acbca^{-1}\},\\
\#12.\ & \{a,b,c \ :\ acab^{-1}c^{-1}ab,\ b^2a^{-1}c^{-1}acb,\ ca^{-1}b^{-1}cabc\},\\
\#13.\ & \{a,b,c \ :\ acab^{-1}c^{-1}ab,\ acbc^{-1}ba^{-1}b,\ b^{-1}abc^2a^{-1}c\},\\
\#14.\ & \{a,b,c \ :\ a^3=[c^{-1},b],\ b^3=a^{-1}cabc^{-1}b^{-1},\ c^3=[a^{-1},b^{-1}][a^{-1},c][b,c].
\end{aligned}
$$

The first three of these were obtained as described above. The last group has no candidate presentations of the kind we have investigated of length up to 21. Its candidate presentation was obtained by considering various presentations whose relators specify that the cube of each generator is a product of commutators.

It is conceivable that finiteness might be proved for some modifications of these presentations or by choosing another enumeration strategy. The variability of coset enumeration performance means that different presentations for the same group may have substantially different behaviour with respect to coset enumeration procedures, as demonstrated in Havas (1991). Further, different enumeration strategies also behave differently.

We also conducted a less-constrained search, where we neither eliminated presentations obviously equivalent to ones already considered nor filtered using p-quotient order. Sometimes we found presentations which performed better under coset enumeration. Further, we found balanced presentations for some groups of order 3^9 and 3^{12}.

The following presentations are for nonisomorphic groups of order 3^9, which are distinct from $M(-2,-2,-2)$. The order of each is readily found by coset enumeration over the trivial subgroup.

#1. $\{a, b, c \ : \ bcb^{-1}ac^{-1}a^2, \ c^{-1}aba^{-1}bcb, \ b^{-1}cbc^2\}$,

#2. $\{a, b, c \ : \ bcb^{-1}ac^{-1}a^2, \ ca^{-1}babc^{-1}b, \ b^{-1}cbc^2\}$.

The following is a presentation for a group of order 3^{12}. It is most readily handled by coset enumeration over the metabelian subgroup $\langle a, c\rangle$ followed by the metabelian quotient calculation for that subgroup.

#1. $\{a, b, c \ : \ b^{-1}c^{-1}ba^2ca, \ c^{-1}abca^{-1}b^2, \ a^{-1}cac^2\}$.

3.5. 5-groups. We investigated presentations of length up to 27 using the techniques described in Section 3.4 and found a number of presentations which passed all filters.

However, in no case could we decide whether these presentations present finite groups. Here we primarily carried out large coset enumerations over particular subgroups.

At least 20 of the 32 isomorphism types of groups of order 5^8 with trivial multiplicator occur among the 5-quotients of the groups presented by these candidate presentations. Here is one example:

$$\{a, b, c \ : \ a^2ca^3b^{-1}c^{-1}b, \ a^{-1}babcb^3c^{-1}, \ cbc^2a^{-1}b^{-1}cac\}.$$

3.6. 2-groups. Recall that the smallest 3-generator 2-groups with trivial multiplicator have order 2^9 and there are just two such groups.

We investigated short presentations and found that both isomorphism types occurred among the 2-quotients of the presentations of length 18 (but not among shorter presentations). It is straightforward using coset enumeration over the trivial subgroup to verify that the following presentations are for groups of order 2^9:

#1. $\{a, b, c \ : \ b^{-1}c^{-1}bca^2, \ c^{-1}bacba^{-1}, \ b^{-1}abc^2a^{-1}\}$,

#2. $\{a, b, c \ : \ ac^{-1}bcab^{-1}, \ acb^2a^{-1}c^{-1}, \ ca^{-1}cb^{-1}ab\}$.

3.7. Non-p-groups. In our less-constrained investigations of presentations, we found various balanced presentations for groups of deficiency zero with order $2^3 3^8$. Here is one example:

$$\{a, b, c \;:\; bcb^{-1}ac^{-1}a^2,\; abcbc^{-1}a^{-1}b,\; b^{-1}abc^2a^{-1}c\}.$$

4. 2-generator groups of deficiency zero

We now apply the methods of the last section to find some p-groups with generator number 2 and deficiency zero.

4.1. Some known examples. There are quite a number of examples of 2-generator p-groups of deficiency zero – see, for example, Johnson (1990, Chapter 7) and Wiegold (1989). We draw attention to just two cases. The presentation

$$\{a, b \;:\; b^{p^{r+s+t}} = a^{p^{r+s}},\; a^b = a^{1+p^r}\}$$

with $r \geq 1, s, t \geq 0$ defines a metacyclic group of order $p^{3r+2s+t}$. For p odd it is routine to see every metacyclic p-group with trivial multiplicator has a presentation as above and the isomorphism type is determined by r, s and t. For the prime 2 there are further examples. Macdonald (1962) showed that, for odd p, the presentations

$$\{a, b \;:\; a^{[a,b]} = a^{1+p},\; b^{[b,a]} = b^{1-p}\}$$

define p-groups and Wamsley (1973) showed the order of these groups is p^5. We will exhibit some other groups of order p^5 with deficiency zero.

4.2. Groups of order p^5 with trivial multiplicator. There are two metacyclic groups of order p^5 with trivial multiplicator. One can deduce using our methods or from the tables of Hall & Senior (1964) that there are no other groups of order 32 with trivial multiplicator.

One can deduce from James (1980) that for the prime 3 there are two non-metacyclic groups of order 3^5 with trivial multiplicator, and for $p \geq 5$ there are $p + 1$.

4.3. Method. We use essentially the same method as that described in Section 3.4 to make a list of presentations, then filter it, divide it into classes according to the isomorphism type of the largest p-quotient and then test for finiteness.

4.4. 3-groups. Keane (1976) showed that the two non-metacyclic 3-groups with trivial multiplicator have deficiency zero. For the record we list presentations obtained with our methods:

#1. $\{a, b \;:\; b^{-2}ababa,\; a^{-2}b^2ab^{-1}ab^2\}$,

#2. $\{a, b \;:\; b^{-2}ababa,\; a^{-1}ba^{-1}ba^2b\}$.

The second defines the Macdonald group.

4.5. 5-groups. We concentrated on obtaining presentations for the six non-metacyclic 2-generator groups of order 5^5. We considered the presentations of length 18 and found four isomorphism types among the 5-quotients of these. We found one additional isomorphism type among the presentations of length 20 and a presentation of length 22 for the sixth group (the Macdonald group).

It is easy to show that the four presentations of length 18 present groups of order 5^5 using coset enumerations over the trivial subgroup. In the two remaining cases, we used coset enumeration over the cyclic subgroup generated by b to obtain the order.

#1. $\{a, b \,:\, b^{-2}aba^3ba,\ a^{-2}b^2abab^2\}$,
#2. $\{a, b \,:\, b^{-1}ab^{-1}aba^2ba,\ a^{-2}b^2abab^2\}$,
#3. $\{a, b \,:\, b^{-2}aba^3ba,\ a^{-1}ba^{-1}bab^2ab\}$,
#4. $\{a, b \,:\, b^{-2}aba^3ba,\ a^{-1}b^3a^{-1}ba^2b\}$,
#5. $\{a, b \,:\, b^{-2}a^2baba^2,\ a^{-1}b^{-1}a^{-1}b^2ab^2ab^2\}$,
#6. $\{a, b \,:\, b^{-1}aba^{-1}b^{-1}a^4ba,\ a^{-1}b^4ab^{-1}a^{-1}bab\}$.

4.6. 7-groups. Again, we focussed on finding presentations for the eight non-metacyclic 2-generator groups of order 7^5. We used coset enumerations over the trivial subgroup or the subgroup generated by b to determine that presentations #1 to #7 present groups of order 7^5. The last is the presentation given by Macdonald – no shorter presentation was found for this group.

#1. $\{a, b \,:\, b^{-2}aba^5ba,\ a^{-1}ba^{-1}b^2ab^2ab^2\}$,
#2. $\{a, b \,:\, b^{-2}aba^5ba,\ a^{-1}b^3a^{-1}b^2a^2b^2\}$,
#3. $\{a, b \,:\, b^{-1}a^2b^{-1}aba^3ba,\ a^{-2}b^2ab^3ab^2\}$,
#4. $\{a, b \,:\, b^{-1}a^3b^{-1}a^2b^2a^2,\ a^{-1}ba^{-1}b^3a^2b^3\}$,
#5. $\{a, b \,:\, b^{-2}a^2ba^3ba^2,\ a^{-1}ba^{-1}b^2ab^2ab^2\}$,
#6. $\{a, b \,:\, b^{-2}aba^5ba,\ a^{-1}b^{-1}a^{-1}b^3ab^2ab^3\}$,
#7. $\{a, b \,:\, b^2a^3b^{-1}ab^{-1}a^3,\ a^2ba^{-3}b^2ab^4\}$,
#8. $\{a, b \,:\, a^{[a,b]} = a^8,\ b^{[b,a]} = b^{-6}\}$.

4.7. Non-p-groups. We have also found groups of deficiency zero with order $2^9 3^5$. Here is one example:

$$\{a, b \,:\, b^{-1}a^{-1}b^{-1}aba^2ba,\ a^{-2}b^2ab^{-1}ab^2\}.$$

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