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Chapter 13

On Sims' Presentation for Lyons' Simple Group

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Abstract

This paper gives a verification that the corrected Sims presentation is indeed a presentation of Lyons simple group. We prove that Lyons original characterization of his new simple group works inside a permutation group of degree 8 835 156.

13.1 Introduction

In his original paper [7], Lyons describes a new finite simple group Ly with the following two properties: Ly has an involution t such that its centralizer $Cen_{Ly}(t)$ is isomorphic to the double cover $2A_{11}$ of the alternating group A_{11}, and additionally $t \notin \mathbf{Z}^*(Ly)$. Sims [11] outlined a presentation for Ly, together with a proof for existence and uniqueness of such a group, but the details have never been published. Nevertheless, others have used this presentation to verify matrix representations for Ly (see e.g. Meyer, Neutsch and Parker [8], and Jansen and Wilson [6]), although the original presentation was not fully correct [6]. Cooperman, Finkelstein, Tselman and York [1] have used probabilistic algorithms and the representation of $Ly \leq GL(111, 5)$ by Meyer, Neutsch and Parker to compute a permutation group G of degree 9 606 125 that should be isomorphic to Lyons simple group Ly. The first author [2] has then used Sims' ideas of bases and strong generating sets [10], together with some lengthy calculations and additional arguments, to prove that G is indeed a group in which the two hypotheses of Lyons' original paper hold.

In this paper, we first recall how to use Sims corrected presentation to produce a permutation group P on 8 835 156 points, where the stabilizer of any point is isomorphic to $G_2(5)$. In particular, we get $|P| = |Ly|$. To check Lyons' original characterization, we have to find the centralizer of an involution in P. To do this, we use the concept of standard generators, as introduced by Wilson [12], as an intermediate step. A recipe to go from Sims' generators to standard generators is given by Jansen and Wilson [6], and we use some results from [2] to find generators for the centralizer as words in the standard generators. To do this, we do some calculations inside the permutation group G of degree 9 606 125, computed by Cooperman, Finkelstein, Tselman and York, using the concept of bases and strong generating sets. During the computations we use some results from [2] to describe our elements, but the actual proof is independent of [2], and can be reproduced with the details given here.

13.2 The Permutation Group of Degree 8 835 156

We start with the corrected Sims presentation as given in [4]. This is actually a presentation on the five generators a, b, c, d, z. The four elements a, b, c, d, together with their relations, generate a group isomorphic to $G_2(5)$ [6, 4]. To get a permutation representation on the cosets of this group we use the Adaptive Coset Enumerator (ACE), developed by the second author. As stated in [4], this run needs roughly half a gigabyte of memory with the parameters chosen. For more details on this enumeration the reader is referred to [4] and the remarks therein. The next lemma states the results of this coset enumeration.

Lemma 13.1. *Sims corrected presentation defines a finite group S with $|S| = |Ly|$. The action of S on the cosets of the subgroup $G_2(5) = \langle a, b, c, d\rangle$ gives a faithful permutation representation for S of degree* 8 835 156.

In the following we use the name P for this permutation group, and we want to show that $P \simeq Ly$. To do this we want to find an involution $t \in P$ such that $Cen_P(t) \simeq 2A_{11}$ and $t \notin \mathbf{Z}^*(P)$. To find generators for the centralizer, we go via so-called standard generators. They have been introduced by Wilson [12] to give a unified treatment in the computational theory of sporadic simple groups. Standard generators are sets of elements that are relatively easy to find, with only a small number of defining relations. In the case of Lyons simple group we have the following from [12].

Proposition 13.1. *Standard generators for Lyons simple group Ly are given by a pair (g_1, g_2) of elements of Ly such that*

$$\begin{aligned} o(g_1) &= 2, \\ g_2 &\in 5A, \\ o(g_1 g_2) &= 14, \\ o((g_1 g_2)^3 g_2) &= 67. \end{aligned}$$

(Note that there are two conjugacy classes of elements of order 5 *in Ly, but we have $(25A)^5 = 5A$, and $25A$ is unique of its order.)*

As mentioned in the introduction, Jansen and Wilson [6] give a way to find standard generators when given Sims' generators. For completeness we state this in the following proposition.

Proposition 13.2. *With Sims' original generators a, b, c, d, z, and the additional element*

$$k_0 = (bzdz)^2 dz,$$

the pair of elements

$$\begin{aligned} k_1 &= (ck_0)^{-8}a^4(ck_0)^8, \\ k_2 &= (ck_0ck_0^2)^{-17}c(ck_0ck_0^2)^{17}. \end{aligned}$$

is a pair of standard generators, if $S \simeq Ly$.

To get to the centralizer of an involution, we now need a way to find generators for such a centralizer as words in the standard generators. This is done in the next section in the permutation representation of Ly on 9 606 125 points.

13.3 The Permutation Group of Degree 9 606 125

In [2] it is shown that the permutation group G of degree 9 606 125, as computed by Cooperman, Finkelstein, Tselman and York, is isomorphic to Lyons simple group. This is done by checking Lyons' characterization from his original paper. To do this, two elements c_3, c_4 are computed that are shown to generate the full centralizer of an involution in G. Unfortunately, standard generators are not introduced in [2], but the first author has made additional calculations in an extended version ([3]) to make compatible the different sets of generators for Ly that are available in the literature. The first step is to go from the two generators of [1], an involution x and an element w of order 67, to standard generators. This is done by the following proposition.

Proposition 13.3. *Given the two generators x and w from [1], the pair of elements*

$$\begin{aligned} s_1 &= x, \\ s_2 &= ((xw)^5)^{xw^{12}xw^{33}} \end{aligned}$$

is a pair of standard generators for $G = \langle x, w \rangle \simeq Ly$.

The next step in [3] is to find a strong generating set for the group generated by s_1 and s_2. This enables us to use Schreier vectors when searching for words for the centralizing elements c_3 and c_4. From [3] we get the following proposition.

Proposition 13.4. *Define*

$$\begin{aligned} s_3 &= (s_2s_1)^2s_2^3s_1(s_2^2s_1)^3s_2^3s_1(s_2^4s_1)^3(s_2^2s_1s_2^4s_1)^2s_2^3, \\ s_4 &= s_3^3(s_1s_3^2)^2s_1(s_3^5s_1s_3^3)^{-1}. \end{aligned}$$

Then the set $\{s_1, s_2, s_3, s_4\}$ is a strong generating set for G with stabilizer chain

$$\{1\} \leq T = \langle s_1, s_4 \rangle \leq H = \langle s_1, s_3 \rangle \leq G.$$

This is enough preparation to write the elements c_3 and c_4 as words in the strong generators s_1, s_2, s_3, s_4, hence as words in the standard generators s_1 and s_2. Having computed the Schreier vectors for the different groups in the stabilizer chain above, we get the following result

Lemma 13.2. *Inside G we have*

$$\begin{aligned} c_3 &= s_1 s_4 s_1 s_4^6 (s_1 s_4)^3 s_4^4 s_1 s_4^3 s_1 s_4 s_3^3 s_1 s_3^2 s_1 s_3 s_1 s_3^3 (s_1 s_3)^2 \cdot \\ &\quad \cdot s_2^2 s_1 (s_2 s_1)^3 s_2^4 s_1 s_2^3 s_1 s_2^2 (s_1 s_2 s_1 s_2^4)^2, \\ c_4 &= s_4^3 s_1 s_4^2 s_1 s_4^3 (s_1 s_4)^4 s_4 s_1 s_4^4 s_3^3 s_1 s_3 s_1 s_3^5 s_1 s_3^3 (s_1 s_3)^3. \end{aligned}$$

This gives a way to find the centralizer of an involution, actually s_1, inside Lyons simple group, when given a pair (s_1, s_2) of standard generators for Ly. The next section deals with the question whether this also works in the permutation group P arising from Sims' group S as described in Section 13.2.

13.4 The final Verification

The standard generators for P, given in Proposition 13.2, together with the words given in Proposition 13.4 and Lemma 13.2, now give us two elements in P, say d_3 and d_4 that should generate the centralizer of the involution g_1, if $P \simeq Ly$. To prove that $P \simeq Ly$, we have to show the following: first we have to prove that $C = \langle d_3, d_4 \rangle$ is the full centralizer of g_1 in P; then we have to show that $C \simeq 2A_{11}$; and finally we need that $g_1 \notin \mathbf{Z}^*(P)$. We begin with the middle part, since this is easy by the calculations done in [2]. There is a classical presentation for $2A_{11}$, already known to Schur [9], given by the following theorem.

Theorem 13.1. *The following are defining relations for the double cover $2A_{11}$ of the alternating group A_{11} as a finitely presented group on the ten generators $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, j$:*

$$\begin{aligned} b_1^3 &= j, & & \\ b_\alpha^2 &= j & & \text{for } \alpha = 2, \ldots, 9, \\ (b_1 b_2)^3 &= j, & & \\ (b_1 b_\lambda)^2 &= j & & \text{for } \lambda = 3, \ldots, 9, \\ (b_\beta b_{\beta+1})^3 &= j & & \text{for } \beta = 2, \ldots, 8, \\ b_\gamma b_\delta &= j b_\delta b_\gamma & & \text{for } \gamma = 2, \ldots, 7 \text{ and } \delta = \gamma + 2, \ldots, 9. \end{aligned}$$

To find words in our two elements d_3 and d_4 for the above relations, we once again quote [2], where another application of Schreier vectors gives such words in the case of the permutation group G of degree 9 606 125. Using the same words inside P, we get the following theorem.

Theorem 13.2. *Define the elements*

$$\begin{aligned} u_0 &= d_3, \\ u_1 &= d_4, \\ u_2 &= u_0^7 u_1 u_0^3 (u_1 u_0)^2 (u_0 u_1)^2 u_0^{-2} u_1 u_0^{-4} (u_1 u_0^{-1})^4 u_0^{-2}, \\ u_3 &= (u_0^2 u_1)^2 u_0 (u_0^2 u_1)^2 u_0 u_1 u_0^{-1} u_1 u_0^{-3} u_1 u_0^{-2} (u_1 u_0^{-1})^3, \\ u_4 &= (u_3 u_2)^2 (u_2 u_3)^2 u_3^2 u_2 u_3^{-1} u_2^{-2} u_3^{-3} u_2^{-1} \end{aligned}$$

as intermediate results. Then the following elements

$$
\begin{aligned}
b_1 &= u_4^4 u_2 u_3^3 u_2^4 u_3 (u_1 u_0)^3 (u_0 u_1)^3, \\
b_2 &= u_4^6 (u_2 u_3)^2 u_3 u_2^5 u_3 (u_0 u_1)^2 u_0^5 (u_1 u_0)^2 u_0^3 u_1, \\
b_3 &= u_4^4 u_3^2 u_2^2 u_3^4 (u_0 u_1)^3 u_0^2 u_1 u_0^5 u_1 u_0 u_1, \\
b_4 &= u_4^4 u_2 u_3 u_2^2 u_3^4 u_2^4 u_1 u_0^6 (u_1 u_0)^3 u_0^5, \\
b_5 &= u_3^2 u_2^2 u_3^3 u_1 u_0^3 (u_1 u_0)^2 u_0^2 u_1, \\
b_6 &= u_4^5 u_2^2 u_3 u_2^4 u_3^2 u_0 u_1 u_0^4 (u_1 u_0^2)^2 u_1 u_0, \\
b_7 &= u_4^2 u_2^3 u_3^3 u_2^2 u_3^2 u_0^6 (u_1 u_0)^2 u_0^3 u_1, \\
b_8 &= u_4^6 u_2^2 u_3^3 u_2 (u_0^2 u_1)^4 u_0^2, \\
b_9 &= u_4^6 u_3^2 u_2^3 u_3 u_2 u_0^6 u_1 u_0^7 u_1 u_0 u_1
\end{aligned}
$$

generate a subgroup D of C that is isomorphic to $2A_{11}$, which is confirmed by checking the relations in Theorem 13.1.

The next step is to prove that $C = D \simeq 2A_{11}$ and that C is the full centralizer of g_1 in P. To do this, we first note that $C \leq Cen_P(g_1)$ by checking the generators d_3 and d_4 of C. To compute the order of $Cen_P(g_1)$, we use the formula for induced characters from page 64 of Isaacs' book on character theory [5]. It is easy to compute that g_1 has 2772 fixpoints in our permutation representation. Now choose any stabilizer $U \simeq G_2(5)$ with $g_1 \in U$. Then U has only one class of involutions, and the abovementioned formula gives

$$\varphi^P(g_1) = \mid Cen_P(g_1) \mid \cdot \frac{\varphi(g_1)}{\mid Cen_U(g_1) \mid}$$

where φ is the trivial character of U. Therefore we get the following lemma.

Lemma 13.3.

$$
\begin{aligned}
\mid Cen_P(g_1) \mid &= \varphi^P(g_1) \cdot \mid Cen_H(g_1) \mid \\
&= 2772 \times 14\,400 = 39\,916\,800 = |2A_{11}|
\end{aligned}
$$

Since $2A_{11} \simeq D \leq C \leq Cen_P(g_1)$, this gives

$$Cen_P(g_1) = C \simeq 2A_{11}$$

To end our proof that Lyons' characterization works for P, we have to show that $g_1 \notin \mathbf{Z}^*(P)$. But we have $g_1 \in U \simeq G_2(5)$, and $\mathbf{Z}^*(G_2(5)) = \{1\}$, hence $g_1 \notin \mathbf{Z}^*(G_2(5))$. This gives the following lemma immediately.

Lemma 13.4.

$$g_1 \notin \mathbf{Z}^*(P)$$

Now we have checked everything from Lyons' characterization, hence conclude with the following final result.

Theorem 13.3. *Sims corrected presentation describes Lyons simple group Ly.*

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Bibliography

[1] G. Cooperman, L. Finkelstein, M. Tselman and B. York: *Constructing Permutation Representations for Matrix Groups*, J. Symbolic Comput. **24**, (1997) 1–18.

[2] H.W. Gollan: *A new existence proof for Ly, the sporadic simple group of R. Lyons*, Preprint **30**, Institut für Experimentelle Mathematik, Universität GH Essen, 1995.

[3] H.W. Gollan: *A Contribution to the Revision Project of the Sporadic Groups: Lyons' simple group Ly*, Vorlesungen aus dem FB Mathematik, volume **26**, Universität GH Essen, 1997.

[4] G. Havas and C.C. Sims: *A presentation for the Lyons simple group*, These proceedings, chapter 14.

[5] I.M. Isaacs: *Character Theory of Finite Groups*,. Academic Press, New York (1976).

[6] C. Jansen and R.A. Wilson: *The minimal faithful 3-modular representation for the Lyons group*, Commun. Algebra **24**, (1996) 873–879.

[7] R. Lyons: *Evidence for a new finite simple group*, J. Algebra **20**, (1972) 540–569.

[8] W. Meyer, W. Neutsch and R. Parker: *The minimal 5-representation of Lyons' sporadic group*, Math. Ann. **272**, (1985) 29–39.

[9] I. Schur: *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, Journal Reine Angew. Math. **139**, (1911) 155–250.

[10] C.C. Sims: *Computational methods in the study of permutation groups*, in: Computational problems in abstract algebra (Pergamon Press, 1970) 169–183.

[11] C.C. Sims: *The existence and uniqueness of Lyons' group*, in: Finite Groups '72 (North Holland, 1973) 138–141.

[12] R.A. Wilson: *Standard generators for sporadic simple groups*, J. Algebra **184**, (1996) 505–515.