Experiences with the Knuth-Bendix procedure

George Havas

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Vaughan-Lee and I proved that 4-Engel groups are locally nilpotent [2] and we wrote about some of the computations that were involved in [3]. These papers cover the relevant background material and include some history of the problem.

A group G is said to be an n-Engel group if

$$[x, \underbrace{y, y, \ldots, y}_{n}] = 1 : \forall x, y \in G.$$

Our proof that 4-Engel groups are locally nilpotent is quite long and complicated. It relies heavily on the result of Traustason [8] that 2-generator 4-Engel groups are nilpotent. A proof of this result of Traustason which uses the Knuth-Bendix procedure was discussed by Charles Sims at this Oberwolfach meeting.

Our proof that 4-Engel groups are locally nilpotent is modelled on the proof used by Vaughan-Lee [10] to show that 4-Engel 5-groups are locally finite. Indeed we started by extending that result to p-groups for primes greater than 5. Vaughan-Lee's proof relies on both hand and machine calculations. In that case the machine work involved use of the p-quotient algorithm and of coset enumeration. To prove the more general result, namely local nilpotence for 4-Engel groups, the previous machine computation tools were no longer directly adequate. Where Vaughan-Lee previously used the p-quotient algorithm to prove properties of p-groups, we use a nilpotent quotient algorithm to prove properties of more general nilpotent groups. Where Vaughan-Lee previously used coset enumeration to prove finiteness of finitely presented groups, we use the Knuth-Bendix procedure to prove nilpotence.

The use of the Knuth-Bendix process as a tool for group theory was pioneered by Charles Sims. The basic procedure (in its application to groups) is described by Gilman [1] and (in great detail) by Sims [7]. A major success of the Knuth-Bendix procedure in group theory was its use by Sims [6] to verify nilpotency of a finitely presented group. The nilpotent quotient algorithm was used initially to construct a polycyclic presentation for the nilpotent quotient Q of the finitely presented group G. Then, using some relations of this presentation as part of an initial set of rewrite rules, and a special term-ordering, Sims developed an effective algorithm for verifying the triviality of the kernel of the quotient Q.

We followed a similar approach to address one particular group which plays a crucial role in our overall proof. This group, which we call T, is a 3-generator 4-Engel group which has nilpotency conditions applying to the subgroups generated by each pair of its given generators. One pair of generators commute, another pair generate a subgroup with class 2, and the remaining pair generate a subgroup with class 3. We start with the following presentation for T:

$$\langle u, v, w \mid [u, v],\ [w, u, u, u],\ [w, u, u, w],\ [w, u, w, w],\ [w, v, v],\ [w, v, w],\ \text{4-Engel} \rangle .$$

T arises as a preimage of a number of groups which we need to prove nilpotent. The proof that those groups are nilpotent follows directly from the fact T is nilpotent. Our computations which prove that T is nilpotent are quite subtle.

Knuth-Bendix implementations well-suited to group theory which are available include `RKBP` by Sims and `KBMAG` by Derek Holt [4] which is available in GAP and MAGMA. Both of these packages may be used in proving enough about T. The final result is that the Knuth-Bendix procedure (initially together with some now surplus "hand" work) shows that the group T is nilpotent.

A simple-minded approach to solving this problem is as follows. Use the nilpotent quotient algorithm (we used the implementation by Werner Nickel [5]) to find a polycyclic presentation (PCP) for the largest nilpotent quotient. Define an input group for Knuth-Bendix on the PCP generators, then run the Knuth-Bendix process using a wreath-product ordering with this input group, adding sufficient instances of the 4-Engel law till the process terminates with a confluent presentation for the group.

This approach works well with easier problems about 3-generator groups. It readily shows the nilpotency of any 3-generator 4-Engel group in which two pairs of generators commute and the remaining pair generate a subgroup with class 3. With more effort, but still easily, it shows the nilpotency of any 3-generator 4-Engel group in which one pair of generators commute and the other two pairs generate a subgroup with class 2. We have not been able to make this approach work directly for T.

When we start this way, we find that the Knuth-Bendix procedure finds many thousands of consequences of the defining relations but does not terminate even when given large amounts of time and space. Various unsuccessful attempts ran for several cpu days and used up to a gigabyte of memory. Instead, our first proof relied on some hand calculations and on our inspecting incomplete Knuth-Bendix computations and deducing useful consequences by hand.

As we did this we discovered two important facts. First, it could be useful to introduce additional, redundant generators. Second, even though the wreath-product ordering implies that the simple approach should lead (eventually) to a finite confluent presentation, we found we could make more rapid progress using lenlex ordering. We point out that the reduction to just the nilpotency problem for the group T came after we had solved similar problems for various other groups. Also, the final proof for T, which requires only one Knuth-Bendix run, was developed after we had done very many other runs.

Our published proof uses one extra generator beyond those which appear in a PCP for (the nilpotent quotient of) T. The Knuth-Bendix procedure then gives us enough additional relations to show that T is nilpotent (with some extra hand calculations). In that proof, we prefaced use of the Knuth-Bendix procedure by determining separately some additional relations which hold in T, beyond the six initial relations in the presentation given for T earlier.

Further study of the relevant computations, aided by suggestions from Charles Sims, reveals that we can use the Knuth-Bendix procedure to prove nilpotence without explicitly adding these extra relations. By adding more redundant generators and by altering the sequence of Knuth-Bendix iterations we can first obtain all of the required additional relations and subsequently deduce that T is nilpotent. It suffices to introduce definitions for all six left normed commutators of weight 5 (instead of just defining one of these, as we did originally). This makes that part of the proof both shorter and faster. Using this would have reduced our complete paper by about 10% in total length, and the proof of the nilpotence of T from about $4\frac{1}{2}$ pages to about 1 page. Furthermore, this proof of the nilpotence of T does not require arguments involving the Hirsch-Plotkin radical.

These observations raise important questions about how one should approach use of the Knuth-Bendix procedure to solve problems in group theory.

It should be noted that Traustason [9] has found a very clever proof of the nilpotence of the group T which does not use the Knuth-Bendix procedure. However both our paper and Traustason's proof of the nilpotence of T still make essential use of detailed information about the free 4-Engel group of rank 2 obtained with the nilpotent quotient algorithm. Also, our complete proof relies on a number of computations using the nilpotent quotient algorithm applied to groups of class 4 and 5.

References

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