Symmetric presentations and orthogonal groups

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Abstract

In this paper we examine series of finite presentations which are invariant under the full symmetric group acting on the set of generators. Evidence from computational experiments reveals a remarkable tendency for the groups in these series to be closely related to the orthogonal groups. We examine cases of finite groups in such series and look in detail at an infinite group with such a presentation. We prove some theoretical results about 3-generator symmetric presentations and make a number of conjectures regarding n-generator symmetric presentations.

1 Introduction

Symmetric presentations have been much studied over a long period; see for example [1, 9, 10, 14, 23], which focus on symmetric relations. An alternative approach, directed towards symmetric generating sets, is taken by Curtis [11, 12, 13].

Suppose G is a finite 2-generator group with generators x_1, x_2 and an automorphism θ with $x_1\theta = x_2$ and $\theta^2 = 1$. Then G has a *symmetric presentation*

$$\mathcal{P} = \{x_1, x_2 \mid w_i(x_1, x_2) = w_i(x_2, x_1) = 1,\ i = 1, 2, \ldots, m\}.$$

Note that any finite non-abelian simple group has such a symmetric presentation. For, if G is a finite non-abelian simple group, then $G = \langle a, b\rangle$ where $a^2 = 1$ [19]. Now consider $H = \langle b, b^a\rangle \leq G$. Either $G = H$ or $|G : H| = 2$. But G is simple, so $G = H = \langle b, b^a\rangle$ and hence G has a symmetric presentation.

Recently Miklos Abert (unpublished) has generalized the results of [14] and conjectures that, given any non-trivial group G, then G has a presentation which, when symmetrized, presents a non-trivial image of G. The above comment on symmetric presentations for simple groups is a consequence of the results of Abert.

Next, for any group G with symmetric presentation $\mathcal{P}$ and any $n \geq 2$, we define the n-generator symmetric presentation

$$\mathcal{P}(n) = \{x_1, x_2, \ldots, x_n \mid w_i(x_j, x_k) = 1,\ i = 1, 2, \ldots, m,\ 1 \leq j \neq k \leq n\}.$$

For example, if $\mathcal{P} = \{x_1, x_2 \mid x_1^3 = x_2^3 = (x_1x_2)^2 = 1\}$ then $\langle\mathcal{P}\rangle = A_4$ and $\langle\mathcal{P}(n)\rangle = A_{n+2}$, the alternating group of degree $n+2$, see [7]. This example was generalized by Sidki [27] (with a relevant correction in [28]) who considered $\mathcal{Q}_t(n)$, where $\mathcal{Q}_t = \{x_1, x_2 \mid x_1^t = x_2^t = (x_1^ix_2^i)^2 = 1,\ 1 \leq i \leq (t-1)/2,\ t \text{ odd}\}$. Sidki showed that the groups $\langle\mathcal{Q}_t(n)\rangle$ are related to orthogonal groups. In this paper we show that orthogonal groups arise as the groups $\langle\mathcal{P}(n)\rangle$ for other symmetric presentations $\mathcal{P}$. Moreover, the nature of $\langle\mathcal{P}(n)\rangle$ varies with increasing n in very diverse ways depending not only on the group $\langle\mathcal{P}\rangle$ but also on the presentation $\mathcal{P}$.

We achieve many of our results by systematic and substantial use of implementations of algorithms. Access to group-theoretic algorithms is provided via the computer algebra systems Cayley [6], GAP [25] and MAGMA [2]; the packages Quotpic [18] and the ANU p-Quotient Program [16, 21]; and various stand-alone programs, see for example [17]. A fundamental tool for computing with finitely presented groups is coset enumeration. We use descendants of the procedure described in [15] which are available in MAGMA and in stand-alone programs. We use Maple for general symbolic calculation. We use the ATLAS [8] as our source of structural information and notation for simple groups.

In section 2 we examine the groups $\langle\mathcal{P}(3)\rangle$ where $\mathcal{P}(2)$ is one of a family of presentations for $L_2(p)$. Section 3 examines 4-generator versions of these groups and shows that both finite and infinite groups can occur. A series of symmetric presentations can collapse to the trivial group as n increases and section 4 investigates two different cases, each starting from a symmetrical presentation of $L_3(3)$. The final section includes a number of conjectures, based on further computational evidence.

2 A family of presentations

We consider the 2-generator symmetric presentation

$$\mathcal{S}_{p,c} = \{x_1, x_2 \mid x_1^p = x_2^p = (x_1^ix_2^{c/i})^2 = 1,\ 1 \leq i \leq p-1\}$$

where p is prime, $c \neq 0$ and division is performed in $GF(p)$.

Beetham [1] showed that $\langle\mathcal{S}_{p,1}\rangle \cong L_2(p)$ for p prime and $p \geq 3$. Sidki [26] showed that it suffices to take $i = 1, 2, 4$ in $\mathcal{S}_{p,1}$. It is easy to see that, for any c, $1 \leq c \leq p-1$, $\langle\mathcal{S}_{p,c}\rangle \cong L_2(p)$ for p prime and $p \geq 3$. The main result of this section is the analysis of $\langle\mathcal{S}_{p,c}(3)\rangle$ for general prime $p \geq 5$. (For $p = 3$ we obtain alternating groups, as shown in [7].)

Theorem 1 *Let* $G = \langle S_{p,c}(3) \rangle$ *(p prime,* $p \geq 5$ *and* $c \neq 0$*) with presentation*

$$\{x, y, z \mid x^p = y^p = z^p = (x^\alpha y^{c/\alpha})^2 = (y^\alpha z^{c/\alpha})^2 = (z^\alpha x^{c/\alpha})^2 = 1,\ \forall \alpha \in GF(p)^*\}.$$

If $\sqrt{c/2} \in GF(p)$, *then* $G \cong O_4^+(p) \cong 2\,(L_2(p) \times L_2(p))$.

Proof: The proof is by way of general hand calculations of the style of those in [4]. The details of the proof involve frequent use of the following lemma.

Lemma 1 *If* $u, v \in G$, $u^p = v^p = 1$, *for p an odd prime, and* $(u^\alpha v^{2/\alpha})^2 = t \in Z(G)$ *for* $\alpha \in \{1, 2\}$, *then* $u^{-1}vu = vuv^{-1}$.

Proof of Lemma 1:

$$\begin{aligned} v^2u^2 &= v^2u^2 \\ \Longrightarrow v(vu^2) &= (v^2u)u \\ \Longrightarrow vu^{-2}v^{-1}t &= u^{-1}v^{-2}ut \end{aligned}$$

Cancel the t's and raise to the power $-1/2 \pmod p$ to get the result. $\square$

The proof of the theorem has the following simple structure. For $\sqrt{c/2} \in GF(p)$ we show (1) that $2\,(L_2(p) \times L_2(p))$ is a homomorphic image of G and (2) that G is a homomorphic image of $2\,(L_2(p) \times L_2(p))$. We use the following presentation for $H = 2\,(L_2(p) \times L_2(p))$:
$\{x_1, x_2, y_1, y_2, t \mid x_i^p = y_i^p = 1,\ [x_1, x_2] = [x_1, y_2] = [y_1, x_2] = [y_1, y_2] = 1,$
$(x_i^\alpha y_i^{c/\alpha})^2 = t,\ t^2 = 1,\ [t, x_i] = [t, y_i] = 1;\ i = 1, 2;\ \alpha \in GF(p)^*\}$.

Observe that it suffices to prove (1) and (2) for $c = 2$. This follows from a consideration of the Tietze transformation of G given by $x \to x^{\sqrt{2/c}}$, $y \to y^{\sqrt{2/c}}$, $z \to z^{\sqrt{2/c}}$ (replacing α by $\sqrt{c/2}\alpha$) and analogous transformations for H.

The details of both steps of the proof are straightforward but tedious. Here we give only the homomorphisms and omit the explicit calculations.

(1) Define $x\phi = x_1y_2$, $y\phi = y_1x_2$, $z\phi = x_1y_1x_1^{-1}x_2y_2x_2^{-1}$.

We claim that $x\phi$, $y\phi$ and $z\phi$ satisfy the relations of G and that the map ϕ is onto.

(2) Define $x_1\psi = (yxz)^2$, $y_1\psi = (zyx)^2$, $x_2\psi = (xyz)^2$, $y_2\psi = (zxy)^2$.

Again we claim that the images of the generators of H satisfy the relations of H and that the map ψ is onto. $\square$

Theorem 2 *Let* $G = \langle S_{p,c}(3) \rangle$ *(p prime,* $p \geq 5$ *and* $c \neq 0$*).*
If $\sqrt{c/2} \notin GF(p)$, *then* $O_4^-(p) \cong L_2(p^2)$ *is a homomorphic image of* G.

Proof: This time we provide mappings to matrix generators. Define
$x\phi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $y\phi = \begin{pmatrix} 1 & 0 \\ -2/c & 1 \end{pmatrix}$, $z\phi = \begin{pmatrix} 1+\sqrt{2/c} & 1 \\ -2/c & 1-\sqrt{2/c} \end{pmatrix}$.

These matrices generate $SL_2(p^2)$ since the images of x and y generate the maximal subgroup $SL_2(p)$ and the image of z is outside this, as $\sqrt{2/c} \notin GF(p)$. Then their images factored by the centre generate $L_2(p^2)$.

Conjecture 1 *Let $G = \langle \mathcal{S}_{p,c}(3)\rangle$ (p prime, $p \geq 5$ and $c \neq 0$). If $\sqrt{c/2} \notin GF(p)$, then $G \cong O_4^-(p) \cong L_2(p^2)$.*

A similar argument to that in the proof of Theorem 1 implies that we only need to prove this for one such c.

To complete this proof it would be sufficient to find a homomorphism from $L_2(p^2)$ onto G. We have not succeeded in doing so, using either matrix generators for $L_2(p^2)$ or a presentation. We have tried to use various presentations from [5] and [29].

We have verified the conjecture for $p \leq 43$ using MAGMA.

3 Four-generator presentations

Definition 1 *Let $X_{\mathcal{P}}(n)$ be the set $\{x_1, \ldots, x_n\}$ of generators in $\langle \mathcal{P}(n)\rangle$. When $\mathcal{P}$ is understood, we simply write $X(n)$.*

The following easy lemma is useful here.

Lemma 2 *If $\langle \mathcal{P}(n-1)\rangle$ is a simple group then either*

(i) $\langle \mathcal{P}(n)\rangle$ is the trivial group, or

(ii) any $n-1$ elements of $X(n)$ generate a subgroup isomorphic to $\langle \mathcal{P}(n-1)\rangle$.

Theorem 3 *Let $G = \langle \mathcal{S}_{5,1}(4)\rangle$. Then $G \cong O_5(5)$.*

Proof: Coset enumeration over $H = \langle x_1, x_2, x_3\rangle$ reveals that this subgroup has index 600 in G. By Theorem 1 and Lemma 2, this shows that $|G| = |O_5(5)|$.

Using the MAGMA system we confirm that the permutations giving the action of G on the cosets of H generate a group isomorphic to $O_5(5)$, which is thus a homomorphic image of G. Together with the value of $|G|$ obtained above, this concludes the proof. □

In contrast, however, we have

Theorem 4 *$\langle \mathcal{S}_{5,2}(4)\rangle$ is an infinite group. It has a homomorphism onto $O_5(4)$, and the kernel of this map has a homomorphism onto an infinite 2-group.*

Proof: Using the low index subgroup algorithm in Cayley we quickly find a subgroup $\langle x_1,\, x_2,\, x_3,\, x_4x_1x_2^{-1}x_3^{-1}x_2^{-1}x_1^{-1}x_4^{-1},\, x_4x_2^{-1}x_3x_4x_2^{-1}x_1^{-1}x_4^{-1}\rangle$ of index 136. Using MAGMA, we show that the permutation action on the cosets of this subgroup generates the group $O_5(4)$ of order 979 200.

Using Quotpic, we construct the homomorphism onto this group, considered as a permutation group on 85 points, and obtain a presentation for the pre-image H of the point stabilizer, $2^6{:}(A_5 \times 3)$. Then, using the low index subgroups algorithm inside Quotpic, we find a subgroup K of index 4 in H, whose core, $\bigcap_{g\in G} K^g$, has index $2^8|O_5(4)|$ in G.

The presentation of K obtained by Quotpic is then simplified by Tietze transformations (also in Quotpic) to obtain a presentation with 3 generators and 17 relations, of total length 265. We then restart Quotpic with this presentation as input.

The abelian quotient of K is cyclic of order 3, and the derived subgroup has a single homomorphism onto A_5 which is found by Quotpic, leading to a presentation of a subgroup L of index 180 in K.

We simplify this presentation, first using the Tietze transformation program in Quotpic and then (to further improve the presentation) the one in GAP, to a presentation with 61 generators, 624 relations and total length 16244. The ANU p-Quotient Program, applied to this presentation, quickly reveals that the largest exponent-2 class-2 quotient of this group has central factors of orders 2^{61} and 2^{1529}. A theorem of M. F. Newman [20] (adjusted for 2-groups as in [22]) then implies that this group must have an infinite 2-quotient.

Remarks: The Cayley run which found the index 136 subgroup also found an index 272 subgroup. A presentation of the index 136 subgroup of G was also obtained and simplified using stand-alone programs to one with 2 generators and 21 relations of total length 253. Subgroups of this group, of index 24, 32 and 64, were then found by low index subgroup methods or random coincidences [2, Example H11E20].

4 Presentations for $L_3(3)$

We consider two symmetric presentations for $L_3(3)$ and observe how they behave differently both from the examples already considered and from each other.

First consider $\mathcal{P}_1 = \{x_1, x_2 \mid x_1^{13} = x_2^{13} = x_1^4x_2x_1x_2^{-3} = x_2^4x_1x_2x_1^{-3} = 1\}$. The group $\langle\mathcal{P}_1\rangle$ is $L_3(3)$, see [3]. $\langle\mathcal{P}_1(3)\rangle$ is trivial, and in fact the same results hold for the corresponding finitely presented semigroup. Proofs of both results are via coset enumerations. For the semigroup given by the presentation $\{x_1, x_2 \mid x_1^{14} = x_1,\ x_2^{14} = x_2,\ x_1^4x_2x_1 = x_2^3,\ x_2^4x_1x_2 = x_1^3\}$ the procedure described in [24] shows that the semigroup is actually the same as the group with that presentation, and the same holds for the analogous 3-generator presentation.

We now consider a member of the parametrized family of symmetric presentations: $\mathcal{T}(p; i_1, i_2; i_3, i_4; \ldots) =$
$\{x_1, x_2 \mid x_1^p = x_2^p = (x_1^{i_1} x_2^{i_2})^2 = (x_2^{i_1} x_1^{i_2})^2 = (x_1^{i_3} x_2^{i_4})^2 = (x_2^{i_3} x_1^{i_4})^2 = \cdots = 1\}$. Let $\mathcal{P}_2 = \mathcal{T}(13; 1, 2; 3, 3; 6, 6)$. Then the group $\langle \mathcal{P}_2 \rangle$ is $L_3(3)$, by coset enumeration.

However, in this case, $\langle \mathcal{P}_2(3) \rangle$ is easily seen to be $L_3(3) \times L_3(3)$, but a direct coset enumeration of $\mathcal{P}_2(4)$ over the trivial subgroup fails. We can however show:

Theorem 5 *The group $\langle \mathcal{P}_2(4) \rangle$ is trivial.*

Proof: A coset enumeration shows that the index $|\langle \mathcal{P}_2(4) \rangle : H \rangle|$, where H is the subgroup generated by three of the symmetric generators, is 1.

Then let G be $\langle \mathcal{P}_2(4) \rangle$, which must be isomorphic to a quotient group of $L_3(3) \times L_3(3)$. If G is not trivial, then G admits a group S_4 of automorphisms acting naturally and faithfully on the generators $X_{\mathcal{P}_2}(4)$. Since the outer automorphism group of $L_3(3) \times L_3(3)$ is dihedral of order 8, it is easy to see that there must be a group $A \cong A_4$ of inner automorphisms of G, also acting naturally on $X(4)$.

Since this is a transitive action the generators are all conjugate in G. The point stabilizer in this action is cyclic of order 3, implying that the centralizer of a generator in G contains a cyclic subgroup of order 39. This implies that $G \not\cong L_3(3)$ since $L_3(3)$ contains no elements of order 39.

Finally, suppose that $G \cong L_3(3) \times L_3(3)$. Elements of order 13 in G have centralizer either $13 \times L_3(3)$ (when we say they are of non-diagonal type) or 13^2 (when we say they are of diagonal type). It is easy to see that each $x \in X(4)$ must be of diagonal type, or else they could not be conjugate and generate the whole group G. This contradicts the existence of an element of order 39 in $C_G(x)$. □

Remarks: The coset enumeration $|\langle \mathcal{P}_2(4) \rangle : \langle x_1, x_2, x_3 \rangle|$ needs a maximum and a total of between ten and twenty thousand cosets using coset table oriented methods. Experience suggests that to enumerate over $\langle x_1, x_2 \rangle$ would require a maximum of about 5000 times as many cosets. This would lead to a much easier proof from a theoretical point of view, but would require a much harder coset enumeration.

Even though both presentations, $\mathcal{P}_1$ and $\mathcal{P}_2$, are based on pairs of generators of order 13 they can be distinguished by observing that in $\mathcal{P}_1$ the generators are in different conjugacy classes while in $\mathcal{P}_2$ they are in the same class.

5 Concluding remarks

For $G_n = \langle S_{5,1}(n)\rangle$, the results of sections 2 and 4 show that:

$$\begin{aligned} G_2 &\cong L_2(5) \cong O_3(5) \\ G_3 &\cong L_2(25) \cong O_4^-(5) \\ G_4 &\cong S_4(5) \cong O_5(5) \end{aligned}$$

Using the computational tools described earlier, we can show that

$$\begin{aligned} G_5 &\cong 2L_4(5) \cong 2O_6^+(5) \\ G_6 &\cong 5^6{:}G_5 \\ G_7 &\cong 2O_8^+(5) \\ G_8 &\cong O_9(5) \end{aligned}$$

Based on this information, we conjecture that

$$\begin{aligned} G_n &\cong O_{n+1}(5), n \equiv 0,2,4(8) \\ G_n &\cong O_{n+1}^-(5), n \equiv 1,3(8) \\ G_n &\cong 2O_{n+1}^+(5), n \equiv 5,7(8) \\ G_n &\cong 5^n{:}G_{n-1}, n \equiv 6(8) \end{aligned}$$

Similarly, for $H_n = \langle S_{5,2}(n)\rangle$, the results of sections 2 and 3 show that $H_2 \cong L_2(5) \cong O_3(5)$, $H_3 \cong 2(L_2(5) \times L_2(5)) \cong O_4^+(5)$ and H_4 is infinite. Note that we can equally well say that $H_2 \cong L_2(4) \cong O_3(4)$ and $H_3 \cong 2(L_2(4) \times L_2(4)) \cong O_4^+(4)$. Furthermore H_4 has a quotient isomorphic to $S_4(4) \cong O_5(4)$. We conjecture that H_n is infinite for $n \geq 4$. Notice that the results of section 4 show that, in certain cases, the group can collapse as n increases, so there is no obvious proof of this last conjecture.

For other values of p similar results hold for $\langle S_{p,c}(n)\rangle$ although for $5 \leq p \leq 23$ both of the series appear to give finite groups for all n, except when $p = 5$ and $\sqrt{c/2} \in GF(5)$ or $p = 7$ and $\sqrt{c/2} \notin GF(7)$.

For $n = 2$, $\langle S_{17,1}(n)\rangle$ and $\langle \mathcal{T}(9;1,2;4,4)(n)\rangle$ are both $L_2(17)$ and their series agree for $2 \leq n \leq 4$, despite having generators with different orders; likewise for $\langle S_{17,3}(n)\rangle$ and $\langle \mathcal{T}(9;1,1;3,-4)(n)\rangle$. It is probable that, were we able to compute these series for larger n, we would find that the four series are all distinct. We can exhibit four distinct series of finite groups containing orthogonal groups over $GF(8)$.

Presentation	n				
	2	3	4	5	6
$\mathcal{T}(9;1,-2;3,4)$	$O_3(8)$	$O_4^-(8)$	$O_5(8)$	$O_6^+(8)$	$O_7(8)$
$\mathcal{T}(9;1,2;3,-4)$	$O_3(8)$	$O_4^+(8)$	$O_5(8)$	$O_6^+(8)$	$O_7(8)$
$\mathcal{T}(9;1,2;(a^2b^{-2}ab^{-2})^2=1)$	$O_3(8)$	$O_4^+(8)$	$O_5(8)$	$O_6^-(8)$	$O_7(8)$
$\mathcal{Q}_7(n)$	$2^6{:}O_2^-(8)$	$O_4^+(8)$	$O_5(8)$	$O_6^-(8)$	$2^{18}{:}O_6^-(8)$

We have computed many other series of groups with symmetric presentations, all containing groups closely related to orthogonal groups. The results are available from the authors.

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